SOLID GEOMETRY

STONE-MILLIS

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SOLID GEOMETRY

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PLANE AND SOLID GEOMETRY

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PREFACE

THE STONE-MILLIS GEOMETRY — PLANE, SOLID, and PLANE AND SOLID — published in 1910, was a pioneer in its field, being the first of the group of American textbooks on geometry which in recent years have attempted in various ways to make the teaching of geometry conform to modern thought in education. It marked a wide departure from the traditional Greek geometry after which textbooks for secondary schools had for generations been patterned. This text has met with remarkable success. The educational ideals which it embodied are now recognized as national, and are summarized in the REPORT OF THE NATIONAL COMMITTEE OF FIFTEEN on the teaching of geometry.

The present geometry, by the same authors, has been prepared in the attempt to produce a text which shall preserve the distinctive features of the older text, but which, if possible, shall be more simple, practical, and teachable.

The following are some of the features which distinguish this text:

I. SIMPLICITY. — 1. The subject is graded so that the easier topics come first and so that the student is introduced to only one new difficulty at a time. The grading of geometry is made possible in this text by abandoning the Greek division of geometry into books and re-grouping the material in chapters.

2. Some of the theorems on fundamental properties of figures are treated informally at the beginnings of many topics.

3. The subject is also abbreviated and simplified by the omission of certain useless traditional theorems and the reduction to a reasonable minimum of the number of theorems, constructions, and corollaries requiring formal treatment.

4. The use of the theory of limits in the proofs of incommensurable cases of theorems has been eliminated.

II. PRACTICALITY. — 1. Geometry is humanized by using as exercises a large number of practical problems. This phase of geometry, which was first introduced into American schools by the Stone-Millis text, is now universally recognized as an integral part of the subject. The STONE-MILLIS GEOMETRY contains a very large number and a very great variety of simple and genuine practical problems. They are selected from many fields of human activity, such as home life, art, architecture, astronomy, engineering, designing, navigation, science, the construction and use of implements and machinery, etc.

2. Directions are given for the construction of many home-made instruments and their use in out-of-door exercises.

3. Geometry is correlated with trigonometry by the introduction of simple work with trigonometric ratios, in the chapter on similar polygons. Application is made to practical problems in the solution of triangles.

III. TEACHABLENESS. — 1. A concrete approach to formal geometry is provided. This develops a body of experience and imagery as a basis of formal geometry, and the latter is not introduced until need for it is felt. In the approach to demonstrative geometry, familiarity with important geometric figures is secured through their accurate construction with drawing instruments. This development of clear imagery through accurate drawing of the figures involved in the formal theorems, etc., is continued throughout plane geometry.

2. Use is made of the suggestive method in the treatment of theorems. While complete model proofs of a large number of theorems are given — and whenever a proof is given it is given in complete form, with numbered steps — the proofs of many theorems are left, with suggestions, to the student. The suggestions in a large part of the theorems are given in the form of analyses. It is believed that suggestions of this nature are superior to those of the traditional kind, which are mere outlines

of the proofs, because they give the student training in exactly the kind of thinking which he must do when attacking a proof unaided, and thus they teach method of attack and develop power of originality.

3. Exercises which demand technical knowledge have been eliminated. Many new exercises have been introduced. The exercises are grouped in every case immediately after the theorem, construction, or corollary to which they relate. Many miscellaneous review exercises are given.

4. Special care has been given to the illustrations in this text. When construction lines are required in drawing a figure, they show in the book. Throughout the SOLID GEOMETRY shaded drawings of models are placed by the side of the more complicated geometric figures, to aid the student in visualizing the third dimension in the figures while looking at the flat drawings of them. The consistent plan of representing hidden parts of figures by thinner lines than the others is carried out, dotted lines being employed exclusively for auxiliary lines as in plane geometry.

Grateful acknowledgment of the authors is due to all those who by timely suggestions have aided in the preparation of this text; especially to Professor H. E. Cobb and Professor A. W. Cavanaugh of Lewis Institute, Chicago; to Miss Alice M. Lord of the High School, Portland, Maine; and to Professor Guido H. Stempel of Indiana University. Special acknowledgment is due to Mr. Charles McCauley of Chicago, who has made the excellent illustrations.

JOHN C. STONE,
JAMES F. MILLIS.

JANUARY, 1916.

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CONTENTS

SYMBOLS

$=$, is equal to; equals.
$\equiv$, is identically equal to.
$\sim$, is similar to.
$\cong$, is congruent to.
$>$, is greater than.
$<$, is less than.
$\doteq$, approaches as a limit.
$\parallel$, is parallel to; parallel.
$\perp$, is perpendicular to; perpendicular.
$\therefore$, therefore.
$\cdots$, and so on.
$\angle$, angle.
$\triangle$, triangle.
▱, parallelogram.
$\odot$, circle.

The plural of any symbol representing a noun is obtained by affixing the letter *s*. Thus, $\angle$s represents angles.

ABBREVIATIONS

Ax., axiom.
Alt., alternate.
Comp., complementary.
Cor., corollary.
Corres., corresponding.
Def., definition.
Ex., exercise.
Ext., exterior.
Hyp., hypothesis.
Int., interior.
Rect., rectangle.
Rt., right.
St., straight.
Supp., supplementary.

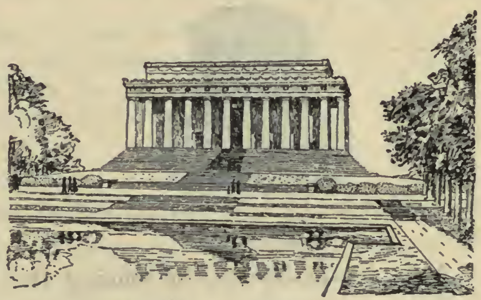
SOLID GEOMETRY

CHAPTER XII

LINES AND PLANES IN SPACE

288. Solid geometry. — It has been seen that *plane geometry* deals with figures which lie in a *flat* or *plane surface*. **Solid geometry** treats of figures consisting of geometric solids, surfaces, lines, points, or combinations of them, which are not confined to a plane.

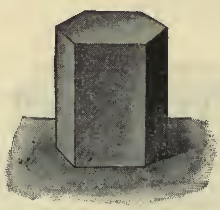
Solid geometry, like plane geometry, has been developed by man in the endeavor to meet his practical needs, and dates back to ancient times. Ahmes, an Egyptian, in his book entitled "Directions for Obtaining the Knowledge of All Dark Things," written about 1700 B.C., gave rules for finding the contents of wells and granaries. The principal geometric forms — prismatic, cylindrical, pyramidal, conical, and spherical — appear in nature in endless combinations. They are employed by man in countless ways — in the construction of buildings, monuments, temples, embankments; in the

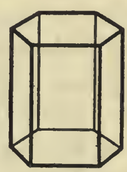


LINCOLN MEMORIAL, WASHINGTON, D. C.

making of pottery, jewelry, furniture; in the great fields of engineering, mechanics, architecture, art, astronomy, etc.

289. Geometric drawings and models. — Although the figures of solid geometry are not in one plane, they must be represented by drawings all parts of which do lie in one plane. It is necessary, therefore, for the student of solid geometry to learn to sense *thickness*, or the *third dimension*, of a figure while looking at a flat drawing of it. For assistance in overcoming this difficulty of the third dimension, those parts of a geometric figure which would be invisible if it were a physical object are represented in a drawing by thinner lines than the other lines of the figure. This is illustrated in the drawing of a prism below. When auxiliary lines occur, they are represented by dotted lines as in plane geometry.



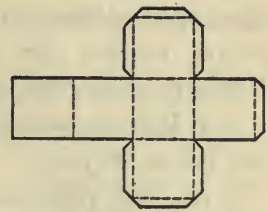


PRISM

Also, for assistance in sensing the third dimension in a drawing, sometimes a picture of a *model* of the figure is given in the text, adjoining the geometric figure, as illustrated in the case of the prism above.

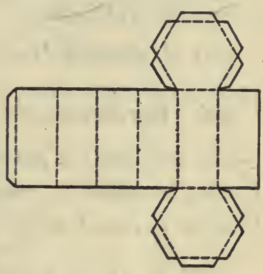
The use of wooden, wire, or cardboard models would be of great aid to the beginning student. Models may be constructed of cardboard as illustrated in the following exercises.

EXERCISES



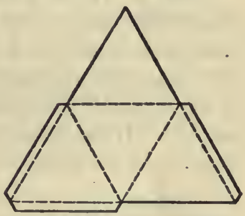
1. On a piece of cardboard, draw a figure similar to the adjoining figure, making the side of each square 3 in. Cut out the pattern, and by folding along the dotted lines and pasting, make a model of a cube.

How many squares form the surface of this model? How many edges has it? How many corners has it?



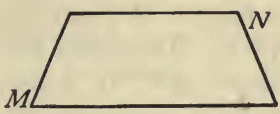
2. On a piece of cardboard, draw a figure similar to the adjoining figure, making the side of each hexagon 2 in. Cut out the pattern, and by folding along the dotted lines and pasting, make a model of a hexagonal prism.

Models of other kinds of prisms may be made by using other kinds of regular polygons instead of the hexagons.



3. Following the method of Exercises 1 and 2, by first making a pattern similar to the pattern here shown, with the side of each triangle 4 in., construct a model of a triangular pyramid.

290. Representation of a plane surface. — A plane surface is unlimited in extent. Hence it is impossible to show an entire plane to the eye by a drawing. In geometric figures, a plane is represented by some kind of polygon, which incloses a portion of the plane. Trapezoids and parallelograms are used extensively for representing planes.



This figure represents a horizontal plane, seen obliquely. The plane is named *MN*.

291. Fundamental property of a plane surface. — A straightedge, held so that it touches a flat or plane surface at two points, touches the surface all along the straightedge. Hence it is inferred that:

(1) *If two points of a straight line lie in a plane, the entire line lies in the plane.*

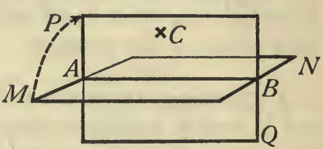
From the fundamental property above, let the student prove the following:

(2) *A straight line can intersect a plane in only one point.*

292. Revolution of a plane. — It is evident that:

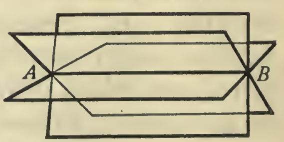
By revolving a plane about any straight line in it as an axis, it may be made to contain any point of space that is not on the line, in one and only one position.

Thus, if AB is a straight line in plane MN and C is any point of space that is not on AB, by revolving MN about AB as an axis, it will arrive at a position PQ in which it contains point C. If the plane is revolved farther, it will no longer contain point C.



293. Corollary 1. — *A straight line lies in an unlimited number of planes.*

For, a straight line AB lies in at least one plane, by definition of a straight line. (See § 6.*) This plane may be revolved about AB, by § 292. And each successive position of the revolving plane represents a different plane containing AB.



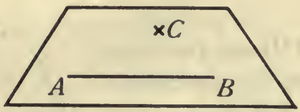
* See the *References to Plane Geometry* at the end of the book.

294. Corollary 2. — *An unlimited number of straight lines may be drawn perpendicular to a given straight line in space at a given point of the line.*

For, at any point of the line AB in § 293, a perpendicular to AB may be drawn in each of the planes containing AB.

295. Determining a plane. — A plane is **determined** by certain points, lines, or combinations of them, if that plane and no other plane contains all of those points, etc.

296. Theorem. — *A straight line and a point not on the line determine a plane.*



Hypothesis. AB is any straight line and C any point of space not on AB.

Conclusion. Line AB and point C determine a plane.

Proof. 1. C is a point not on straight line AB. Hyp.

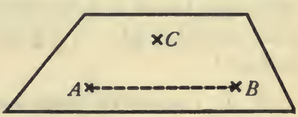
2. AB lies in a plane. § 6

3. By revolving this plane about AB as an axis, it may be made to contain point C in one and only one position. § 292

4. Hence line AB and point C determine a plane. § 295

Without the book, draw a figure and write out this proof.

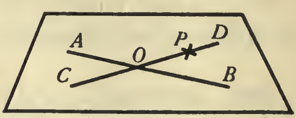
297. Corollary 1. — *Three points not in the same straight line determine a plane.*



For, if A, B, and C are the three points, A and B determine a straight line. One and only one plane can contain line AB and point C, by § 296. Hence, one and only one plane can contain A, B, and C. Why?

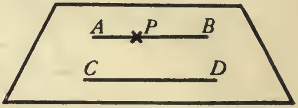
298. Corollary 2. — *Two intersecting straight lines determine a plane.*

For, if the straight lines AB and CD intersect at O, let P be any other point on CD. Then one and only one plane can contain AB and point P, by § 296. This plane contains the whole line CD, by § 291. Why can only this one plane contain the lines AB and CD?



299. Corollary 3. — *Two parallel straight lines determine a plane.*

For, if AB and CD are parallel, they lie in a plane, by definition of parallel lines. Let P be any point on AB. Then only one plane can contain point P and CD. Why? Why, then, can only one plane contain AB and CD?



EXERCISES

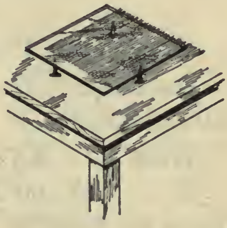
1. What tool does a carpenter use for reducing the rough surface of a board to a smooth or plane surface? Explain how the principle in § 291 is involved in the use of this tool.

2. A plasterer, after putting plaster on a wall, smooths it down to a plane surface by rubbing a board or straightedge over it in many directions. Builders of concrete sidewalks do the same thing. Why does this produce a plane surface?

3. In measuring grain with a half-bushel measure, the grain is first heaped, then it is raked off even with the top of the measure, with a board. Why is the measure then *level* full? What kind of surface is thus formed by the grain?

4. Illustrate the principle in § 292 by using the point of a pencil to denote any point of space and a page of an open book to denote the revolving plane. Think of another way in which this principle may be illustrated.

5. Place three tacks on a table so that they are not in a straight line and have their points up. Support a book or a pane of glass on the points of the tacks. How does this illustrate § 297?



6. Place a ruler on a table, with its edge up, and place a tack on the table, with its point up. Support a book or a pane of glass on the edge of the ruler and the point of the tack. How does this illustrate § 296?

7. Place two rulers in parallel positions on a table, with their edges up. Support a book or pane of glass on the edges of the rulers. How does this illustrate § 299?

8. Cameras, telescopes, etc., are mounted on tripods. Will an object mounted on three legs always rest firmly on the floor? Why?

9. Do four given points lie in one plane?

10. Why do not all chairs, tables, etc., with four legs rest firmly on the floor?

11. Find four points in the room through which a plane may be passed. Find four points through which a plane cannot be passed.

12. Hold two pencils in such positions as to show that a plane cannot be passed through any two straight lines taken at random.

13. Show that if a piece of cord is fastened at both ends, and, by grasping it at any third point, the two parts are stretched straight, the stretched cord will lie entirely in one plane.

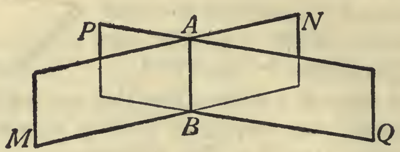
14. Show that any transversal of two parallel lines must lie in the plane of those lines.

15. Show that a figure made of three straight lines, each intersecting the other two, but not in a common point, must lie in one plane.

300. Intersection of planes. — It is assumed that:

If two planes intersect, they have more than one point in common.

301. Theorem. — *The intersection of two planes is a straight line.*



Hypothesis. Planes MN and PQ intersect.

Conclusion. The intersection of planes MN and PQ is a straight line.

Proof. 1. MN and PQ intersect. Hyp.

2. ∴ MN and PQ have at least two common points. Let them be A and B. § 300

3. Then all points of the straight line joining A and B lie in both planes. § 291

4. No point without line AB can lie in both planes, for the two planes would then coincide. § 296

5. Hence, since all common points of the planes, and no other points, must lie in the intersection, line AB is the intersection of the planes. That is, the intersection of the planes is a straight line.

Write out the proof without using the book.

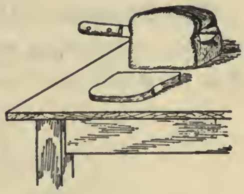
EXERCISES

1. What is the meaning of "Two intersecting planes determine a straight line"?

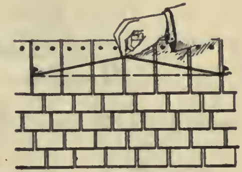
2. What is the locus of all points common to two planes?

3. Show that if a sheet of cardboard, or other kind of stiff sheet material, is folded and creased, the crease must be straight.

4. When one saws a board in two, why is the edge of the cut made by the saw a straight line?

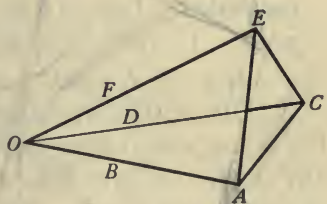


5. In slicing a loaf of bread, if a surface of the loaf is flat, why is the edge of the slice straight? Think of other applications in the home of the principle in § 301.



6. Carpenters, in shingling a roof, often mark off straight chalk lines on the roof to assist in placing the shingles in straight rows. A cord, whitened with chalk, is held firmly at the two ends, and stretched straight. Then grasping the cord at some third point, they pull it away a little from the roof and let it go. It springs back and strikes the roof, leaving a white mark, which is a straight line. Prove that this is an application of § 301.

7. When three planes, not containing the same straight line, intersect in pairs, if the three lines of intersection are not parallel, they meet at one point.



SUGGESTIONS. — Let the planes intersect in AB, CD, and EF. Prove that AB and CD intersect at some point O. Prove that O is on EF by showing that it is in plane EC and also in plane AE.

8. Illustrate the principle in Exercise 7 by the walls and ceilings of the schoolroom. Give other illustrations.

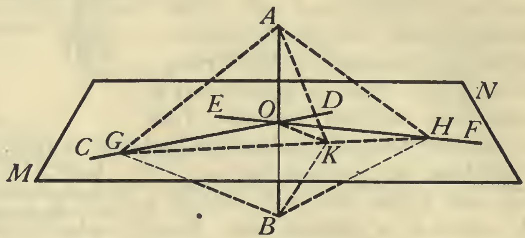
302. Line and plane perpendicular. — A straight line which intersects a plane is **perpendicular to the plane** if it is perpendicular to every straight line in the plane drawn through the point of intersection.

The plane also is said to be **perpendicular to the line.**

If a line is perpendicular to a plane, the point of intersection of the line and plane is called the **foot** of the perpendicular.

If a straight line intersects a plane, but is not perpendicular to it, the line is said to be **oblique to the plane.**

303. Theorem. — *If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane determined by them.*



Hypothesis. Lines CD and EF intersect at O, and determine plane MN; $AB \perp CD$ and $AB \perp EF$ at O.

Conclusion. Line AB is perpendicular to plane MN.

Proof. 1. In plane MN, CD and EF intersect at O; also $AB \perp CD$ and $AB \perp EF$ at O. Hyp.

2. Draw OK, any other straight line through O in MN. Draw any straight line intersecting CD at G, EF at H, and OK at K. On AB, take B on the opposite side of O from A so that $OB = AO$. Draw AG, AH, AK, BG, BH, BK.

3. Then $AG = BG$ and $AH = BH$. § 105

4. GH is a common side of $\triangle AGH$ and $\triangle BGH$.

5. $\therefore \triangle AGH \cong \triangle BGH$. § 76

6. $\therefore \angle AGH = \angle BGH$. Def. congruence

7. In $\triangle AGK$ and $\triangle BGK$, GK is a common side.

8. $\therefore \triangle AGK \cong \triangle BGK$. § 63

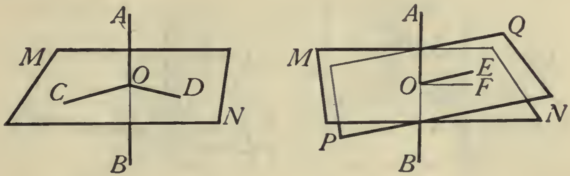
9. $\therefore AK = BK$. Def. congruence

10. $\therefore AB \perp OK$. § 107

11. Hence, since OK is any other line than CD and EF drawn through O in MN, AB is perpendicular to the plane MN. § 302

Write the proof in full without referring to the book.

304. Theorem. — *Through a given point in a given straight line, one plane and only one plane can be drawn perpendicular to the line.*



Hypothesis. AB is a given straight line, and O is a given point on AB.

Conclusion. Through O, one plane and only one plane can be drawn perpendicular to AB.

Proof. 1. AB is a given straight line, and O is a given point on AB. Hyp.

2. Two straight lines, OC and OD (figure at left), can be drawn perpendicular to AB at O. § 294

3. OC and OD determine a plane MN. § 298

4. Plane $MN \perp$ line AB. § 303

5. Suppose that MN is not the only plane through O perpendicular to AB, and that PQ (figure at right) is another plane through O perpendicular to AB.

6. Then a plane can be drawn containing AB and intersecting planes MN and PQ in two straight lines OF and OE, respectively. § 301

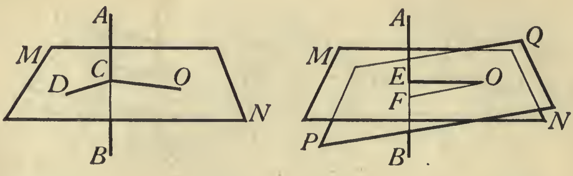
7. $OF \perp AB$ and $OE \perp AB$. § 302

8. But this is impossible. § 53

9. $\therefore$ the supposition is false, and hence MN is the only plane through O, perpendicular to AB.

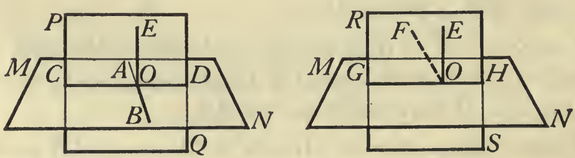
Draw figures, and write the proof in full without consulting the book.

305. Theorem. — *Through a given point without a given straight line, one plane and only one plane can be drawn perpendicular to the line.*



Suggestions. If AB (figure at left) is the line and O the point, O and AB determine a plane, in which OC can be drawn $\perp AB$. Then another line CD can be drawn $\perp AB$. OC and CD determine a plane $MN \perp AB$. Show (figure at right) by using § 54 that no other plane PQ can be drawn through $O \perp$ line AB.

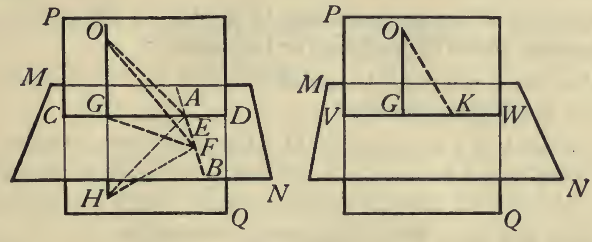
306. Theorem. — *Through a given point in a given plane, one line and only one line can be drawn perpendicular to the plane.*



Suggestions. If O (figure at left) is a point in plane MN, straight line AB can be drawn through O in MN: then plane PQ can be drawn through $O \perp AB$, intersecting MN in CD; then line OE can be drawn in $PQ \perp CD$. Then $OE \perp$ plane MN.

Show (figure at right) by using § 53 that no other line OF can be drawn through $O \perp$ plane MN.

307. Theorem. — *Through a given point without a given plane, one line and only one line can be drawn perpendicular to the plane.*



Hypothesis. MN is a plane and O a point not in MN.

Conclusion. Through O one line and only one line can be drawn perpendicular to MN.

Proof. 1. MN is a plane and O a point not in MN. Hyp.

2. Let AB be any straight line in MN (figure at left).

3. A plane PQ can be drawn through $O \perp AB$. § 305

4. PQ intersects MN in a straight line CD, and line AB at a point E. § 301

5. Draw OG in $PQ \perp CD$. Let GF be any straight line drawn in MN, meeting AB at F. Produce OG through G to H, making $GH = OG$. Draw OE, OF, HE, HF.

6. $AB \perp OE$ and $AB \perp HE$. § 302

7. $\therefore \angle OEF$ and $\angle HEF$ are rt. $\angle$s. Def. $\perp$

8. $OE = HE$. § 105

9. In $\triangle OEF$ and $\triangle HEF$, EF is a common side.

10. $\therefore \triangle OEF \cong \triangle HEF$. § 64

11. $\therefore OF = HF$. Def. congruence

12. $\therefore GF \perp OG$. § 107

13. $\therefore OG$ is perpendicular to plane MN. § 303

The proof that OG is the only line that can be drawn through O perpendicular to MN is left to the student. Use the figure at the right. Employ § 54.

EXERCISES

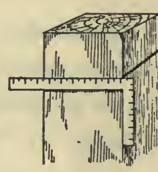
1. Explain how to test by means of a carpenter's square whether or not a post on a level surface is vertical.

2. Establish a line perpendicular to the top of the table by using two rectangular pieces of cardboard or two books.

3. How can a carpenter determine if a floor is level by means of a plumb line and a steel square?

4. Explain how a carpenter could set a timber perpendicular to the floor by using only a ten-foot pole notched at the six-foot and eight-foot points.

5. Show how by § 303 a carpenter is enabled to saw a piece of lumber squarely in two.



6. Prove that any point in the plane perpendicular to a given line-segment at its middle point is equidistant from the ends of the line-segment.

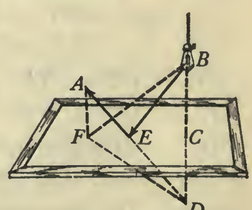
7. Prove that any point of space equidistant from the ends of a given line-segment is in the plane perpendicular to the line-segment at its middle point.

8. Prove that if a plane is perpendicular to a line-segment at its middle point, any point not in the plane is unequally distant from the ends of the line-segment.

9. What is the locus of points in space equidistant from two given points?

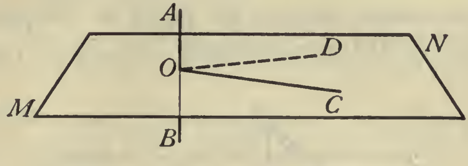
10. Find the locus of points in a given plane that are equidistant from any two given points not in that plane.

11. A ray of light that starts from a given point B and is reflected by a mirror to a given point A travels along the shortest possible path. Find the point of the mirror from which the ray is reflected.



SUGGESTION. — Draw a perpendicular from one of the given points to the plane of the mirror, and extend it to a point an equal distance on the other side of the plane of the mirror. Join this point to the other given point. Prove that the point at which this second line meets the surface of the mirror is the required point.

308. Theorem. — *All lines perpendicular to a given line at a given point lie in the plane perpendicular to the given line at the given point.*



Hypothesis. Plane MN is perpendicular to line AB at O; OC is any line perpendicular to AB at O.

Conclusion. OC lies in MN.

Suggestions. Suppose that OC does not lie in MN. Then the plane determined by AB and OC must intersect MN in another line OD. Show that this violates § 53.

Write the proof in full.

EXERCISES

1. If a spoke of a wheel is perpendicular to the axle upon which it turns, it describes a plane in its rotation.

2. The locus of all lines perpendicular to a given line at a given point is the plane perpendicular to the given line at the given point.

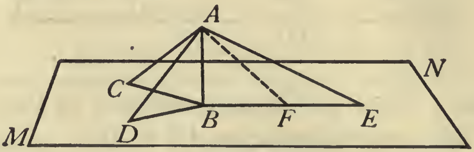
3. If a plane is perpendicular to a given line, any line perpendicular to the given line through any point of the given plane must lie in the given plane.

309. Theorem. — *The perpendicular to a plane from an external point is the shortest line-segment that can be drawn to the plane from the point.*

The proof is left to the student. Write the proof in full.

The perpendicular line-segment drawn from a given point to a given plane is called the **distance** from the point to the plane.

310. Theorem. — (1) *Oblique line-segments drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular from the point, are equal;* (2) *of two oblique line-segments meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Hypothesis. Line AB is perpendicular to plane MN, meeting MN at B; oblique line-segments from A meet MN at C, D, E; $BD = BC$ and $BE > BC$.

Conclusion. $AD = AC$, and $AE > AC$.

Suggestions. It may be proved that $AD = AC$ if it is proved that $\triangle ABD \cong \triangle ABC$.

For proving $AE > AC$, on BE mark off $BF = BC$, and draw AF. Point F lies between B and E. Why? It can be proved that $AE > AC$ if it is shown that $AC = AF$ and $AE > AF$. But $AE > AF$ if $\angle AFE > \angle AEF$, etc.

Write the proof in full.

311. Theorem. — (1) *Equal oblique line-segments drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular from the point;* (2) *of two unequal line-segments, the greater meets the plane at the greater distance from the foot of the perpendicular.*

Suggestions. This theorem, which is the converse of the theorem in § 310, may be proved easily by the indirect method, making use of § 310.

The proof is left to the student. Write the proof in full.

EXERCISES

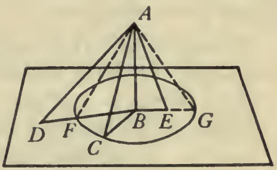
1. If a derrick stands vertically on level ground, guy ropes reaching from the top of the derrick to stakes in the ground at equal distances from the foot of the derrick are equal.

2. If a line-segment AB is perpendicular to a plane at B, it subtends equal angles at all points of a circle lying in the plane and having B as center.

3. If the line-segment AB is perpendicular to a plane, and intersects the plane at a point C on AB produced, AB subtends equal angles at all points of a circle lying in the plane and having C as center.

4. If the line-segment AB is perpendicular to a plane, and intersects the plane at a point C between A and B, AB subtends equal angles at all points of a circle lying in the plane and having C as center.

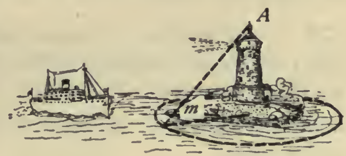
5. If a line-segment AB is perpendicular to a plane at B, and a circle is drawn in the plane, with center at B, then the angle subtended by AB at a point of the plane within the circle is greater, and at a point without the circle less, than the angle subtended at a point on the circle.



SUGGESTION. — Prove $\angle AEB > \angle ACB$ and $\angle ADB < \angle ACB$.

6. If a line-segment AB is perpendicular to a plane at B, and a circle is drawn in the plane, with center at B, then a point of the plane is within, on, or without the circle, according as AB subtends at that point an angle greater than, equal to, or less than the angle which it subtends at a point on the circle.

7. A ship may be steered past a region of danger by observing the angle of elevation at the ship subtended by a landmark A, within this region, as follows: A map contains a circle with the foot of A as center, and large enough to inclose the region of danger. The size of the angle m which the landmark A subtends at a point of the circle is noted on the map. The course of the ship is so directed that the angle of elevation of A as observed from the ship from time to time never becomes greater than angle m. Prove that the ship does not enter the region of danger.

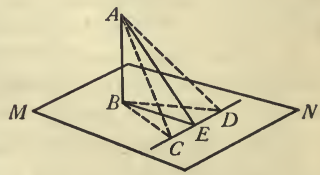


8. If the ceiling of a room is ten feet high, how can a point of the floor that is directly under a given point of the ceiling be located by using only an 11-foot pole, chalk, and a string?

9. What is the locus of the feet of equal line-segments drawn to a plane from a point in a perpendicular to the plane? Prove it.

10. The locus of points each of which is equidistant from all points of a circle is a straight line through the center and perpendicular to the plane of the circle.

11. If, from the foot of a perpendicular to a plane, a line is drawn perpendicular to any given line in the plane, the line which joins the point of intersection to any point in the perpendicular to the plane is perpendicular to the given line in the plane.



SUGGESTIONS. — Let AB be perpendicular to plane MN; let CD be any line in MN; let $BE \perp CD$. It is required to prove $AE \perp CD$.

Mark off $CE = DE$, and draw BC, BD, AC, AD.

It can be proved that $AE \perp CD$ if it is first proved that $AC = AD$. It can be proved that $AC = AD$ if it is first proved that $\triangle ABC \cong \triangle ABD$; etc.

12. Prove the converse of Exercise 11, *i.e.*, if $AB \perp$ plane MN and $AE \perp CD$, then $BE \perp CD$.

SUGGESTIONS. — Mark off $CE = DE$ and draw the auxiliary lines AC, BC, AD, and BD as in the proof of Exercise 11.

It may be proved that $BE \perp CD$ if what is proved first? How may the latter be proved?

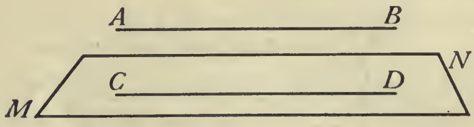
13. If a given line is perpendicular to a given plane, all of the lines which are perpendicular to a line of the plane from points of the given line are concurrent.

SUGGESTION. — In the figure of Exercise 11, AE intersects CD at what special point?

312. Parallel lines and planes. — A straight line and a plane which do not intersect are said to be **parallel**.

Two planes which do not intersect are called **parallel planes.**

313. Theorem. — *If a straight line not in a given plane is parallel to a line in the plane, it is parallel to the plane.*



Hypothesis. Line CD lies in plane MN; line $AB \parallel CD$, and AB is not in MN.

Conclusion. AB is parallel to plane MN.

Proof. 1. Line CD lies in plane MN; line $AB \parallel CD$, and AB is not in MN. Hyp.

2. $\therefore$ AB and CD are in a plane. § 299

3. This plane intersects plane MN in CD, because CD is in both planes.

4. Hence, if AB met MN, the point of intersection would be on CD, because it would be in both planes.

5. But AB cannot meet CD. Def. $\parallel$

6. $\therefore$ AB cannot meet MN.

7. $\therefore$ AB is parallel to MN. § 312

EXERCISES

1. State the converse of the theorem in § 313. Is it a true theorem? Illustrate.

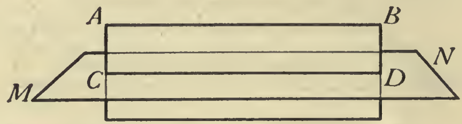
2. Through a given point without a given straight line, an unlimited number of planes can be passed which are parallel to the given line.

3. If two straight lines are not in the same plane, a plane can be drawn containing one of them and parallel to the other.

4. Through a given point without a given plane, any number of lines can be drawn parallel to the plane.

5. If two intersecting planes are drawn through two parallel lines, their line of intersection is parallel to each of the lines.

314. Theorem. — *If a straight line is parallel to a plane, the intersection of this plane and any plane containing the given line is parallel to the given line.*



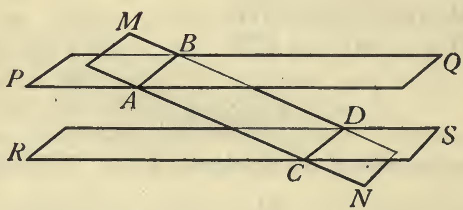
Hypothesis. Line AB is parallel to plane MN; plane AD contains AB, and intersects plane MN in line CD.

Conclusion. $CD \parallel AB$.

Suggestion. If AB and CD intersect, AB must intersect plane MN, which is impossible.

Write the proof in full.

315. Theorem. — *If a plane intersects two parallel planes, the lines of intersection are parallel.*



Hypothesis. Plane PQ is parallel to plane RS; plane MN intersects plane PQ in AB and plane RS in CD.

Conclusion. $AB \parallel CD$.

Suggestion. AB and CD lie in the same plane and cannot intersect. Why?

Write the proof in full.

EXERCISES

1. Segments of parallel lines which are cut off by two parallel planes are equal.

2. If a line is parallel to a plane, a plane can be drawn containing the given line and parallel to the given plane.

3. If a straight line and plane are parallel, any straight line drawn through a point of the plane and parallel to the given line lies wholly in the given plane.

SUGGESTIONS. — The two parallel lines determine a plane, which intersects the given plane in a line through the given point.

4. If a line is parallel to each of two intersecting planes, it is parallel to their line of intersection.

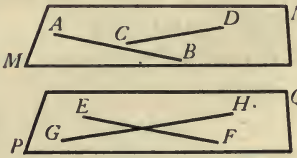
5. If a straight line is parallel to a plane, any straight line which is parallel to the given line and not in the given plane is also parallel to the given plane.

SUGGESTION. — The two parallel lines determine a plane. Consider the two cases when this plane is parallel to the given plane and when it intersects it. In the latter case, what is known about the line of intersection?

6. If each of two intersecting straight lines is parallel to a given plane, the plane determined by these lines is parallel to the given plane.

SUGGESTION. — Use indirect proof. Apply § 314, then § 46.

7. If two lines of one plane are parallel respectively to two intersecting lines of another plane, the planes are parallel.



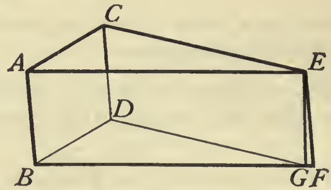
SUGGESTIONS. — If $AB \parallel EF$ and $CD \parallel GH$, what relation has plane MN to EF and to GH? Now apply Exercise 6.

8. Through a given point a plane can be passed parallel to any two given straight lines in space.

SUGGESTION. — Through the given point, draw lines parallel to the given lines.

9. If a straight line is parallel to each of two given planes, the lines of intersection which any plane passing through the given line makes with the given planes are parallel.

316. Theorem. — *Two straight lines which are parallel to a third straight line not in their plane are parallel to each other.*



Hypothesis. $CD \parallel AB$ and $EF \parallel AB$.

Conclusion. $CD \parallel EF$.

Proof. 1. $CD \parallel AB$ and $EF \parallel AB$. Hyp.

2. AB and CD determine a plane AD, and AB and EF determine a plane AF. § 299

3. CD and point E determine a plane CG. § 296

4. Planes CG and AF intersect in a line EG. § 301

5. $CD \parallel$ plane AF. § 313

6. $\therefore CD \parallel EG$. § 314

7. $AB \parallel$ plane CG. § 313

8. $\therefore AB \parallel EG$. § 314

9. $\therefore EG$ and EF coincide. § 46

10. $\therefore CD \parallel EF$. Step 6

Write out the proof without the book.

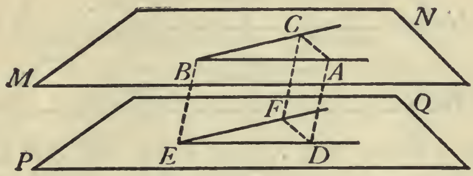
EXERCISES

1. If a straight line and plane are parallel, the lines of intersection of the given plane and all planes which contain the given line are parallel.

2. In a rectangular room the intersection of the ceiling and a wall is parallel to the intersection of the floor and the opposite wall.

3. In a quadrilateral whose sides are not all in the same plane (called a *gauche* quadrilateral), the line-segments which join the middle points of the adjacent sides form a parallelogram.

317. Theorem. — *If two angles in different planes have their sides parallel each to each and extending in the same direction from the vertices, the angles are equal and their planes are parallel.*



Hypothesis. $\angle ABC$ is in plane MN, and $\angle DEF$ is in plane PQ; $BA \parallel ED$ and $BC \parallel EF$, and the parallel lines extend in the same directions from the vertices.

Conclusion. $\angle ABC = \angle DEF$ and $MN \parallel PQ$.

Proof. 1. $BA \parallel ED$ and $BC \parallel EF$. Hyp.

2. Mark off on the sides of the angles $BA = ED$ and $BC = EF$. Draw BE, AD, CF, AC, DF.

3. Then $EDAB$ and $EFCB$ are parallelograms. § 90

4. $\therefore AD \parallel BE$ and $CF \parallel BE$. Def. ▱

5. $\therefore AD \parallel CF$. § 316

6. Also $AD = BE$ and $CF = BE$. § 82

7. $\therefore AD = CF$. Ax. I

8. $\therefore ADFC =$ ▱. § 90

9. $\therefore AC = DF$. § 82

10. $\therefore \triangle ABC \cong \triangle DEF$. § 76

11. $\therefore \angle ABC = \angle DEF$. Def. congruence

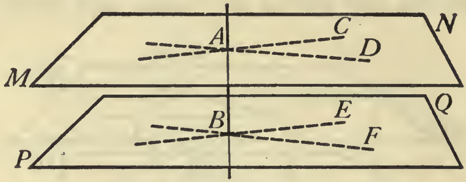
12. Now $BA \parallel$ plane PQ and $BC \parallel$ plane PQ. § 313

13. If MN and PQ intersected, their line of intersection would meet either BA or BC or both. § 46

14. But this is impossible. § 312

15. $\therefore MN$ and PQ do not intersect; and therefore $MN \parallel PQ$. § 312

318. Theorem. — *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Hypothesis. Plane $MN \parallel$ plane PQ; line $AB \perp$ plane PQ.

Conclusion. Line $AB \perp$ plane MN.

Proof. 1. Plane $MN \parallel$ plane PQ; and line $AB \perp$ plane PQ. Hyp.

2. Let two planes containing AB intersect MN in AC and AD, respectively, and PQ in BE and BF, respectively.

3. Then $AC \parallel BE$ and $AD \parallel BF$. § 315

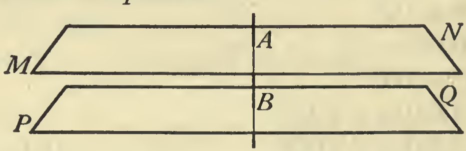
4. But $AB \perp BE$ and $AB \perp BF$. § 302

5. $\therefore AB \perp AC$ and $AB \perp AD$. § 34

6. $\therefore$ line $AB \perp$ plane MN. § 302

Draw a figure and write the proof without the book.

319. Theorem. — *Two planes which are perpendicular to the same straight line are parallel.*



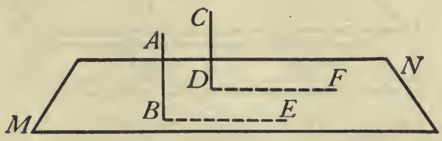
Hypothesis. Plane $MN \perp$ line AB at A; plane $PQ \perp$ line AB at B.

Conclusion. $MN \parallel PQ$.

Suggestion. Suppose MN and PQ not parallel, and from any point of their line of intersection, lines drawn to A and B.

Write the proof in full.

320. Theorem. — *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



Hypothesis. Lines AB and CD meet plane MN at B and D, respectively; $AB \perp$ plane MN; $AB \parallel CD$.

Conclusion. $CD \perp$ plane MN.

Proof. 1. $AB \perp$ plane MN and $AB \parallel CD$. Hyp.

2. In MN draw any line DF from D, and draw $BE \parallel DF$.

3. Then $\angle ABE = \angle CDF$. § 317

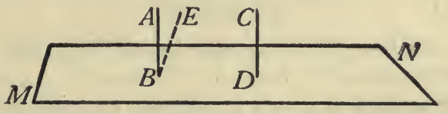
4. But $AB \perp BE$. § 302

5. $\therefore \angle ABE =$ rt. $\angle$. Def. $\perp$

6. $\therefore \angle CDF =$ rt. $\angle$. Ax. I

7. Hence, since DF is *any* line in MN through D, $CD \perp$ plane MN. § 302

321. Theorem. — *Two lines perpendicular to the same plane are parallel to each other.*



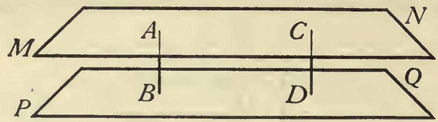
Hypothesis. Line $AB \perp$ plane MN; line $CD \perp$ plane MN.

Conclusion. $AB \parallel CD$.

Suggestion. Suppose AB not parallel to CD, but $EB \parallel CD$. Apply § 320. Write out the complete proof.

322. Distance between parallel planes. — A line-segment connecting points in two parallel planes and perpendicular to each plane (See § 318) is called the **distance** between the parallel planes.

323. Theorem. — *Two parallel planes are everywhere equally distant.*



Hypothesis. Plane $MN \parallel$ plane PQ.

Conclusion. MN and PQ are everywhere equally distant.

Suggestion. Let AB and CD be any two line-segments between MN and PQ, each being perpendicular to both planes. Prove $AB = CD$. Write the proof in full.

EXERCISES

1. If three equal line-segments which are not in one plane are parallel, the triangles formed by joining their corresponding end points are congruent and lie in parallel planes.

2. Two points on the same side of a plane and equally distant from it determine a line that is parallel to the plane.

3. Through a given point only one plane can be passed parallel to a given plane.

4. Two planes each parallel to a third plane are parallel to each other.

5. A straight line and a plane, both perpendicular to the same straight line, are parallel.

6. What is the locus of points equidistant from two given parallel planes? Prove it.

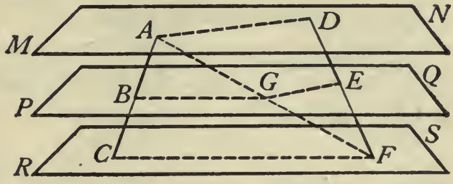
7. What is the locus of points equidistant from any two given points and also equidistant from two given parallel planes? Prove it.

8. What is the locus of points at a given distance from a given plane? Prove it.

9. Find a point equidistant from two given points, equidistant from two given parallel planes, and at a given distance from a third given plane.

10. If two angles in different planes have their sides parallel each to each, one pair of parallel sides extending in the same direction and the other pair in opposite directions from the vertices, the angles are supplementary.

324. Theorem. — *If two straight lines intersect three parallel planes, their corresponding segments cut off by the planes are proportional.*



Hypothesis. Planes MN, PQ, and RS are parallel, and cut off segments AB and BC from line AC, and corresponding segments DE and EF from line DF.

Conclusion. $\dfrac{AB}{BC} = \dfrac{DE}{EF}$.

Suggestions. Draw AF, intersecting PQ at G, and draw BG, CF, AD, and GE. The conclusion may be established if it is first proved that $\dfrac{AB}{BC}$ and $\dfrac{DE}{EF}$ are each equal to $\dfrac{AG}{GF}$. This may be proved if it is first proved that $BG \parallel CF$, etc.

Write the proof in full.

EXERCISES

1. A line meets three parallel planes in the points A, B, and C, respectively. A second line meets the planes in the corresponding points D, E, and F, respectively. $AB = 6$ in., $BC = 8$ in., and $DF = 18$ in. Find the lengths of DE and EF.

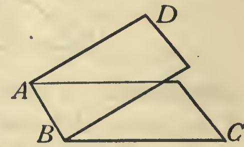
2. The distances between four shelves are 6 in., 8 in., and 10 in., respectively. A diagonal brace 36 in. long reaches from the top shelf to the bottom shelf. Find the segments of the brace between the shelves, making no allowance for the thickness of the shelves.

3. If three parallel planes intercept equal segments on one straight line, they intercept equal segments on any other straight line which they intersect.

4. If two straight lines intersect any number of parallel planes, their corresponding segments are proportional.

325. Diedral angles. — When two planes intersect, the line of intersection divides each plane into two parts. One such part of one plane and one such part of the other plane together form a figure called a **diedral angle.**

The parts of planes forming a diedral angle are called the **faces** of the diedral angle, and the line of intersection of the planes is called the **edge** of the diedral angle.

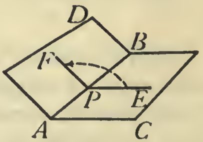


Thus, in the figure, the planes AC and BD are the faces, and the line AB is the edge of the diedral angle.

A diedral angle is named by naming a point in one face, then the edge, then a point in the other face. When confusion would not result, a diedral angle may be named by merely naming its edge.

Thus, the diedral angle above is named "angle C–AB–D," or merely "angle AB."

326. The plane angle of a diedral angle. — The angle formed by straight lines drawn in the two faces of a diedral angle, perpendicular to the edge at the same point of the edge, is called the **plane angle** of the diedral angle.

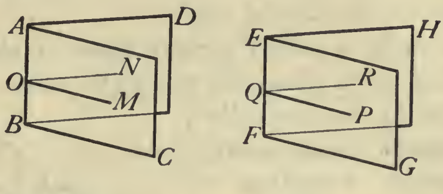


Thus, if $PE \perp AB$ and PE lies in face BC, and if $PF \perp AB$ and PF lies in face AD, of diedral angle C–AB–D, then $\angle EPF$ is the plane angle of diedral angle C–AB–D.

It is evident from § 317 that the plane angle of a diedral angle has the same size at whatever point of the edge its vertex is taken.

Let the student draw a figure representing any two positions of the plane angle of a diedral angle, and reason out this truth.

327. Theorem. — *Two diedral angles are equal if their plane angles are equal.*



Hypothesis. $\angle MON$ and $\angle PQR$ are plane angles of diedral angles $C–AB–D$ and $G–EF–H$, respectively; and $\angle MON = \angle PQR$.

Conclusion. Angle $C–AB–D$ = angle $G–EF–H$.

Proof. 1. $\angle MON$ and $\angle PQR$ are plane angles of diedral angles $C–AB–D$ and $G–EF–H$, respectively; and $\angle MON = \angle PQR$. Hyp.

2. $\therefore \angle MON$ can be superposed on $\angle PQR$ so that they coincide. Let it be so placed. § 13

3. $AB \perp OM$, $AB \perp ON$; also $EF \perp QP$, $EF \perp QR$. § 326

4. $\therefore AB \perp$ plane of $\angle MON$, and $EF \perp$ plane of $\angle PQR$. § 303

5. But plane of $\angle MON$ and plane of $\angle PQR$ coincide. § 298

6. $\therefore AB$ and EF coincide. § 306

7. $\therefore$ the faces of angle $C–AB–D$ and the faces of angle $G–EF–H$ coincide. § 298

8. $\therefore$ angle $C–AB–D$ = angle $G–EF–H$, because they coincide throughout.

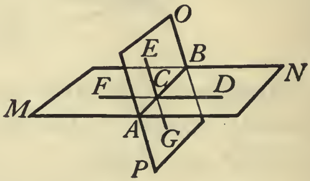
328. Theorem. — *If two diedral angles are equal, their plane angles are equal.*

The proof of this theorem is left to the student. Use superposition, as in § 327. Write the proof in full.

329. Special diedral angles. — It follows from § 327 and § 328 that the plane angle of a diedral angle may be taken as the *measure* of the diedral angle.

A diedral angle is called a **right, acute**, or **obtuse** diedral angle according as its plane angle is right, acute, or obtuse.

Two diedral angles are called **adjacent, vertical, complementary,** or **supplementary,** according as their plane angles are adjacent, vertical, etc.



If two planes are intersected by a third plane, the terms *corresponding diedral angles*, etc., are applied to the pairs of diedral angles in the same sense that the corresponding terms are applied to plane angles.

EXERCISES

1. Name a pair of vertical diedral angles in the figure of § 329. Name a pair of adjacent diedral angles. Name a pair of supplementary diedral angles.

2. Draw a figure representing two complementary diedral angles.

3. Prove that any two vertical diedral angles are equal.

4. Prove that any two right diedral angles are equal.

5. Prove that diedral angles which are complements of the same diedral angle or equal diedral angles are equal.

6. Prove that diedral angles which are supplements of the same diedral angle or equal diedral angles are equal.

Prove that if two parallel planes are cut by a third plane:

7. The corresponding diedral angles formed are equal.

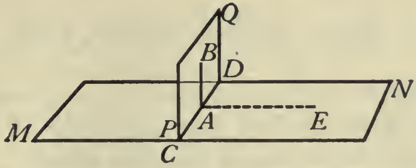
8. The alternate interior diedral angles formed are equal.

9. The alternate exterior diedral angles formed are equal.

10. The consecutive interior diedral angles formed are supplementary.

330. Perpendicular planes. — Two intersecting planes which form a right diedral angle are called **perpendicular planes.**

331. Theorem. — *If a straight line is perpendicular to a plane, any plane which contains the line is perpendicular to the plane.*



Hypothesis. Line $AB \perp$ plane MN at A; plane PQ contains AB.

Conclusion. Plane $PQ \perp$ plane MN.

Proof. 1. Line $AB \perp$ plane MN at A; plane PQ contains AB. Hyp.

2. $\therefore PQ$ and MN intersect in a straight line CD. § 301

3. Draw $AE \perp CD$ in plane MN.

4. $AB \perp CD$. § 302

5. Then $\angle EAB$ is the plane angle of diedral angle $N\text{–}CD\text{–}Q$. § 326

6. But $AB \perp AE$. § 302

7. $\therefore \angle EAB =$ rt. $\angle$. Def. $\perp$

8. $\therefore$ plane $PQ \perp$ plane MN. § 330

332. Theorem. — *A straight line drawn in one of two perpendicular planes, perpendicular to their line of intersection, is perpendicular to the other plane.*

Suggestion. If perpendicular planes MN and PQ intersect in line CD, and line $AB \perp CD$ and lies in PQ, draw line $AE \perp CD$ in plane MN. Then the analysis is as follows: $AB \perp$ plane MN if $AB \perp AE$. And $AB \perp AE$ if $\angle EAB$ = rt. $\angle$. And $\angle EAB =$ rt. $\angle$ if $\angle EAB$ is the plane angle of diedral angle $N\text{–}CD\text{–}Q$ and angle $N\text{–}CD\text{–}Q$ is a right diedral angle, etc.

Write the proof in full.

333. Corollary 1. — *If two planes are perpendicular, a line perpendicular to one of them at any point in their intersection lies in the other.*

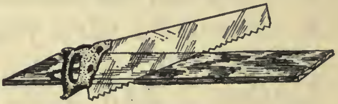
Use indirect proof. Apply § 306 and § 332.

334. Corollary 2. — *If two planes are perpendicular, a line perpendicular to one of them from any point of the other that is not in the intersection lies in the other plane.*

Use indirect proof. Apply § 307 and § 332.

EXERCISES

1. In sawing a board in two with a polished handsaw, the image of the board is seen in the surface of the saw. When the image of the surface of the board and the surface of the board itself appear to form one plane, the saw is cutting the board squarely in two, *i.e.* at right angles. Why?



2. The edge of a diedral angle is perpendicular to the plane of its plane angle.

3. If a plane is perpendicular to the intersection of two planes, it is perpendicular to each of the planes.

4. The plane of the plane angle of a diedral angle is perpendicular to the faces of the diedral angle.

5. If a plane is perpendicular to a line in another plane, it is perpendicular to that plane.

6. Through a given straight line not perpendicular to a given plane, one and only one plane can be passed perpendicular to the given plane.

7. If three lines are perpendicular to each other at a common point, the planes of the lines are perpendicular to each other.

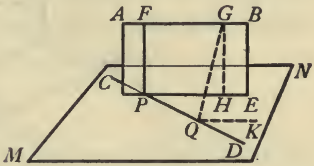
8. If a line and a plane are parallel, any plane perpendicular to the line is also perpendicular to the plane.

9. If three or more planes intersect in a common line, the lines perpendicular to them from any common external point lie in one plane.

10. If two planes are perpendicular to each other, a line perpendicular to one of the planes and not contained in the other is parallel to the other.

11. If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

12. Between two straight lines not in the same plane, one, and only one common perpendicular can be drawn..

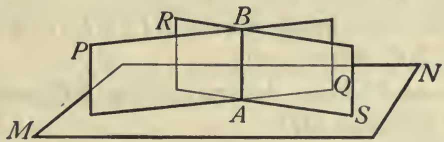


SUGGESTIONS. — Let AB and CD be the given lines. Pass plane MN through CD parallel to AB. Through AB, pass a plane AE perpendicular to MN, intersecting MN in PE and CD at P. In plane AE draw $PF \perp PE$. Prove PF a common perpendicular to AB and CD.

Suppose that GQ is a second common perpendicular, draw $GH \perp PE$, draw $QK \parallel AB$, and show that an absurdity results.

13. The common perpendicular between two lines not in the same plane is the shortest line-segment that can be drawn between the lines.

335. Theorem. — *If each of two intersecting planes is perpendicular to a third plane, their line of intersection is perpendicular to that plane.*



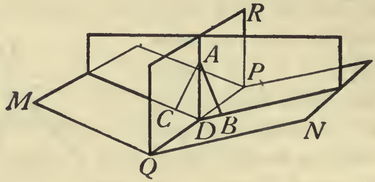
Hypothesis. Plane $PQ \perp$ plane MN; plane $RS \perp$ plane MN; and planes PQ and RS intersect in line AB, which meets MN at A.

Conclusion. Line $AB \perp$ plane MN.

Suggestions. Draw line $AC \perp$ plane MN. Then $AB \perp$ plane MN if AB and AC are proved to coincide. AB and AC coincide if AC is proved to lie in PQ and in RS. Hence begin by proving the latter.

Write the proof in full.

336. Theorem. — *Every point in the plane which bisects a diedral angle is equally distant from the faces of the diedral angle.*



Hypothesis. Plane QR bisects diedral angle $M\text{–}QP\text{–}N$; A is any point in QR; line $AB \perp$ plane NP and line $AC \perp$ plane MP.

Conclusion. $AB = AC$.

Proof. 1. Plane QR bisects diedral angle $M\text{–}QP\text{–}N$; A is any point in QR; line $AB \perp$ plane NP and line $AC \perp$ plane MP. Hyp.

2. AB and AC determine a plane. § 298

3. This plane intersects planes NP, QR, and MP in lines DB, DA, and DC, respectively. § 301

4. The plane which is determined by AB and AC is perpendicular to NP and MP. § 331

5. $\therefore$ the plane which is determined by AB and AC is perpendicular to edge QP. § 335

6. $\therefore DB \perp QP$, $DA \perp QP$, $DC \perp QP$. § 302

7. $\therefore \angle BDA$ and $\angle ADC$ are plane angles of diedral angles $N\text{–}QP\text{–}R$ and $R\text{–}BP\text{–}M$, respectively. § 326

8. $\therefore \angle BDA = \angle ADC$. § 328

9. In $\triangle ABD$ and $\triangle ACD$, AD is common, and $\angle ABD$ and $\angle ACD$ are right angles. § 302

10. $\therefore \triangle ABD \cong \triangle ACD$. § 68

11. $\therefore AB = AC$. Def. congruence

337. Theorem.—*Any point within a diedral angle, and equally distant from its faces, lies in the plane which bisects the diedral angle.*

Suggestion. If A is a point within diedral angle M–QP–N, equally distant from MP and NP, let QR be the plane determined by A and edge QP. Prove that plane QR bisects diedral angle M–QP–N. Write the proof in full.

EXERCISES

1. Any point not in the bisecting plane of a diedral angle is unequally distant from its faces.

2. The locus of points within a diedral angle and equidistant from the faces is the bisecting plane of the diedral angle.

3. Find the locus of points equidistant from two given points and also equidistant from the faces of a diedral angle.

4. Find the locus of points equidistant from two parallel planes and also equidistant from the faces of a diedral angle.

5. Find the locus of points at a given distance from a given plane and also equidistant from the faces of a given diedral angle.

6. Find the locus of points equidistant from the faces of a given diedral angle and also equidistant from the faces of another given diedral angle.

7. Find a point in a given plane that is equidistant from the faces of a given diedral angle and also at a given distance from a given plane.

8. Find a point in a given plane that is equidistant from two given points and also equidistant from the faces of a given diedral angle.

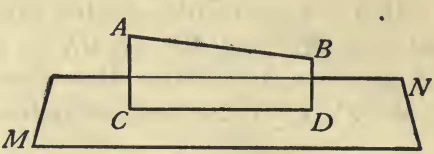
9. The bisecting plane of any diedral angle, if produced through the edge, also bisects the vertical diedral angle.

10. If two planes intersect, forming two adjacent diedral angles, the planes which bisect these adjacent angles are perpendicular to each other.

338. Projections. — The **projection of a point** upon a plane is the foot of the perpendicular from the point to the plane.

The **projection of a given line** upon a plane is the line containing the projections upon the plane of all points of the given line.

339. Theorem. — *The projection of a straight line upon a plane is the straight line determined by the projections of any two of its points upon the plane.*



Hypothesis. A and B are any two points of straight line AB; C and D are the projections of A and B, respectively, upon plane MN.

Conclusion. The projection of AB upon MN is straight line CD.

Proof. 1. C and D are projections of A and B, respectively, upon MN. Hyp.

2. $\therefore AC \perp$ plane MN and $BD \perp$ plane MN. § 338

3. $\therefore AC \parallel BD$. § 321

4. $\therefore AC$ and BD determine a plane AD. § 299

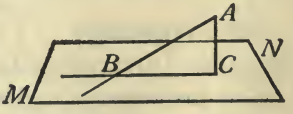
5. Planes AD and MN intersect in straight line CD. § 301

6. Plane $AD \perp$ plane MN. § 331

7. $\therefore CD$ contains the projections upon MN of all points of AB. § 334

8. $\therefore$ the projection of AB upon MN is straight line CD. § 338

340. Inclination of a line to a plane. — A straight line which intersects a plane intersects its projection upon the plane. Why?



The acute angle formed by a straight line which intersects a plane, but is not perpendicular to it, and its projection upon the plane is called the **inclination of the line to the plane**, as $\angle CBA$.

EXERCISES

1. Show that in the special case when a line is perpendicular to a plane, its projection upon the plane is a point.

2. If a trough is placed directly beneath the edge of a roof, all water dripping from the edge of the roof falls into it. Show how this illustrates the theorem in § 339.

3. If a line-segment is parallel to a plane, its projection upon the plane is a line-segment equal to the given line-segment.

4. A line-segment 12 in. long has an inclination of 30° with a plane. Find the length of its projection upon the plane. Find its length if the inclination is 45°. If it is 60°.

5. If a straight line is not perpendicular to a plane, the given line and its projection upon the plane determine a second plane perpendicular to the first.

Suggestion. — From a point on the given line, draw a line perpendicular to the given plane. Does the plane determined by the given line and its projection contain this perpendicular?

6. The projections upon a plane of two parallel lines which are not perpendicular to the plane are parallel.

Suggestion. — From a point on each of the given parallel lines draw a line perpendicular to the given plane. What relation have these perpendiculars? Seek to apply § 317.

7. If two parallel lines intersect a plane, they are equally inclined to it.

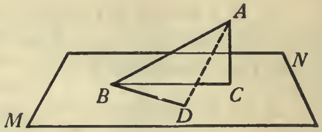
8. Parallel line-segments are proportional to their projections upon a plane.

9. If a straight line intersects two parallel planes, it is equally inclined to the two planes.

10. If two planes are not perpendicular, the projection upon one of any parallelogram in the other is a parallelogram.

11. If two planes are parallel, the projection upon one of any polygon in the other is a congruent polygon.

12. If a straight line intersects a plane, but is not perpendicular to it, the inclination of the line to the plane is the least angle made by the given line and any line drawn in the plane through its foot.



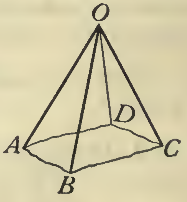
SUGGESTION. — Let BC be the projection of BA, and C the projection of A, upon MN; and let BD be any other line in MN than BC that passes through B. Mark off $BD = BC$ and draw AD. Prove $\angle CBA < \angle DBA$.

13. If a line intersects a plane, and is not perpendicular to it, the obtuse angle which it makes with its projection upon the plane is the greatest angle that it makes with any straight line drawn in the plane, through the intersection.

SUGGESTION. — In the figure of Ex. 12, produce BC and BD through B.

341. Polyedral angles. — The figure formed by three or more rays which are drawn from the same point, not more than two being in one plane, together with the portions of planes determined by the pairs of adjacent rays and included between them, is called a **polyedral angle.**

The rays are called the **edges** of the polyedral angle. The point from which the rays are drawn is called the **vertex.** The portions of planes determined by pairs of adjacent rays and included between them are called the **faces.** And the angles formed in the faces by the pairs of adjacent edges are called the **face angles** of the polyedral angle.

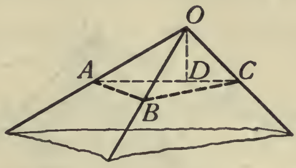


A polyedral angle whose vertex is O and whose edges are OA, OB, OC, and OD is named $O\text{–}ABCD$.

A polyedral angle is **convex** if a plane which intersects all of the faces intersects them in line-segments which form a convex polygon.

A polyedral angle of three faces is called a **triedral angle.**

342. Theorem. — *Any face angle of a triedral angle is less than the sum of the other two face angles.*



Hypothesis. $\angle AOC$ is the greatest face angle of triedral angle $O\text{–}ABC$.

Conclusion. $\angle AOC < \angle AOB + \angle BOC$.

Proof. 1. $\angle AOC$ is the greatest face angle of triedral angle $O-ABC$. Hyp

2. Draw OD in face AOC, making $\angle AOD = \angle AOB$. And let a plane cut off $OD = OB$ and meet OC at C and OA at A.

3. AO is a common side of $\triangle AOD$ and $\triangle AOB$.

4. $\therefore \triangle AOD \cong \triangle AOB$. § 63

5. $\therefore AD = AB$. Def. congruence

6. But $AC < AB + BC$. § 146

7. $\therefore DC < BC$. Ax. VII

8. $\therefore \angle DOC < \angle BOC$. § 148

9. $\therefore \angle AOD + \angle DOC$ or $\angle AOC < \angle AOB + \angle BOC$. Ax. VII

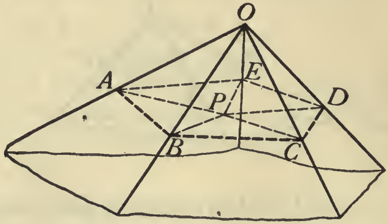
EXERCISES

1. Any face angle of a polyedral angle is less than the sum of all of the other face angles.

Suggestion. — Draw planes through any one edge of the angle and all of the other non-adjacent edges. Apply § 342.

2. Any face angle of a triedral angle is greater than the difference between the other two face angles.

343. Theorem. — *The sum of the face angles of any convex polyedral angle is less than four right angles.*



Hypothesis. $O\text{–}ABC \cdots$ is a convex polyedral angle.

Conclusion. $\angle AOB + \angle BOC +$ etc. < 4 rt. $\angle$s.

Proof. 1. $O\text{–}ABC \cdots$ is a convex polyedral angle. Hyp.

2. Let a plane meet the edges at A, B, C, etc. Let P be any point within polygon $ABC \cdots$. Draw AP, BP, CP, etc.

3. Then $\angle OBA + \angle OBC > \angle PBA + \angle PBC$,

$\angle OCB + \angle OCD > \angle PCB + \angle PCD$, etc.

§ 342

4. $\therefore$ by adding, $\angle OBA + \angle OBC + \angle OCB +$ etc. $>$ $\angle PBA + \angle PBC + \angle PCB +$ etc. Ax. IX

That is, the sum of the base angles of the triangles whose common vertex is O is greater than the sum of the base angles of the triangles whose common vertex is P.

5. But the sum of the angles of each triangle is a st. $\angle$.

§ 48

6. $\therefore$ since the number of triangles whose vertex is O equals the number of triangles whose vertex is P, the sums of all angles of the two sets are equal. Ax. II

7. $\therefore$ the sum of the angles whose vertex is O is less than the sum of the angles whose vertex is P. Ax. VIII

8. But the sum of the angles at $P = 4$ rt. $\angle$s. § 17

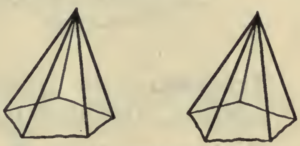
9. $\therefore \angle AOB + \angle BOC +$ etc. < 4 rt. $\angle$s. Ax. XII

EXERCISES

1. The sum of the face angles of a triedral angle is 320°. What is the greatest value that any one of the face angles may have?

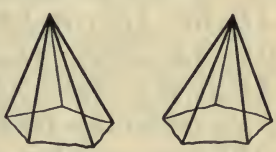
2. If the face angles of a triedral angle are equal, what is the greatest value that each of the face angles may have?

344. Equal and symmetrical polyedral angles. — Two polyedral angles are **equal** if they may be made to coincide. Hence to each face angle or diedral angle in one of two equal polyedral angles, there corresponds a like angle equal to it in the other, and these corresponding parts are arranged in the *same order*.



EQUAL POLYEDRAL ANGLES

Two polyedral angles are called **symmetrical** if to each face angle or diedral angle in one there corresponds a like angle equal to it in the other, and these corresponding parts are arranged in *reverse order*. In general, symmetrical polyedral angles cannot be made to coincide.



SYMMETRICAL POLYEDRAL ANGLES

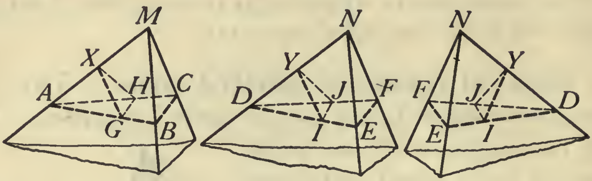
Two symmetrical polyedral angles may be compared to a pair of gloves. To each part of one glove there corresponds a like part of the other, but these like parts are arranged in *reverse order*, so that the right hand cannot be put into the left glove.

EXERCISES

1. Construct from cardboard, or stiff paper, models of two symmetrical triedral angles whose face angles are respectively 60°, 45°, and 30°. Cut as indicated in the pattern, fold along dotted lines, and paste. Can they be made to coincide?

2. If the faces of a polyedral angle are produced through the vertex, the given angle and the vertical polyedral angle formed are symmetrical.

345. Theorem. — *If two triedral angles have the face angles of one equal respectively to the face angles of the other, their corresponding diedral angles are equal.*



Hypothesis. In triedral angles $M\text{–}ABC$ and $N\text{–}DEF$, $\angle AMB = \angle DNE$, $\angle BMC = \angle ENF$, $\angle AMC = \angle DNF$.

Conclusion. Angle $B\text{–}AM\text{–}C =$ angle $E\text{–}DN\text{–}F$, etc.

Proof. 1. $\angle AMB = \angle DNE$, etc. Hyp.

2. Mark off equal distances MA, MB, MC, ND, NE, NF, on the edges of angle $M\text{–}ABC$ and angle $N\text{–}DEF$. Draw AB, BC, AC, DE, EF, DF. On AM and DN, respectively, take $AX = DY$. Draw $XG \perp AM$ in face AMB, meeting AB at G; $XH \perp AM$ in face AMC, meeting AC at H; $YI \perp DN$ in face DNE, meeting DE at I; $YJ \perp DN$ in face DNF, meeting DF at J. Draw GH and IJ.

3. Then $\triangle AMB \cong \triangle DNE$, etc. § 63

4. $\therefore AB = DE$, and $\angle BAM = \angle EDN$, etc. Def. cong.

5. $\therefore \triangle ABC \cong \triangle DEF$. § 76

6. $\therefore \angle BAC = \angle EDF$. Def. cong.

7. Also $\triangle AGX \cong \triangle DIY$ and $\triangle AHX \cong \triangle DJY$. § 67

8. $\therefore XG = YI$, $XH = YJ$; also $AG = DI$, $AH = DJ$. Def. cong.

9. $\therefore \triangle AGH \cong \triangle DIJ$. § 63

10. $\therefore GH = IJ$. Def. cong.

11. $\therefore \triangle GXH \cong \triangle IYJ$. § 76

12. $\therefore \angle GXH = \angle IYJ$. Def. cong.

(To be completed by the student.)

346. Theorem. — *If two triedral angles have the face angles of one equal respectively to the face angles of the other, and arranged in the same order, the triedral angles are equal.*

Suggestion. Use superposition. Show that by placing the triedral angles so that one pair of equal face angles coincide, the triedral angles may be made to coincide throughout. Apply § 345. Write out the complete proof.

EXERCISES

1. If two triedral angles have the face angles of one equal respectively to the face angles of the other, but arranged in reverse order, the triedral angles are symmetrical.

2. If two triedral angles have two face angles and the included diedral angle of one equal respectively to two face angles and the included diedral angle of the other, and the equal parts arranged in the same order, the two triedral angles are equal.

Suggestion. — Superpose one triedral angle upon the other.

3. If two triedral angles have two diedral angles and the included face angle of one equal respectively to two diedral angles and the included face of the other, and arranged in the same order, the two triedral angles are equal.

4. If two triedral angles have two face angles and the included diedral angle of one equal respectively to two face angles and the included diedral angle of the other, but the equal parts arranged in reverse order, the triedral angles are symmetrical.

Suggestion. — Construct a third triedral angle symmetrical to one of the given triedral angles. Compare the parts of the three angles.

5. If two triedral angles have two diedral angles and the included face angle of one equal respectively to two diedral angles and the included face angle of the other, but the equal parts arranged in reverse order, the triedral angles are symmetrical.

6. If two face angles of a triedral angle are equal, the diedral angles opposite them are equal.

7. If two face angles of a triedral angle are equal, the triedral angle is equal to the symmetrical triedral angle.

MISCELLANEOUS EXERCISES

1. In how many positions must a spirit level be observed in order to determine if the surface on which it rests is horizontal? Why?

2. A room is 10 ft. high, 16 ft. wide, and 20 ft. long. Find the length of the shortest line that can be drawn on the floor and walls from a lower corner to the diagonally opposite upper corner.

3. Can a triedral angle have for its faces a regular decagon and two equilateral triangles? Two regular octagons and a square? Why?

4. If the inclination of a line-segment to a plane is 45° and the projection upon the plane is 16 in. long, how long is the line-segment?

5. If each of a series of parallel planes intersects all faces of a triedral angle, the intersections form a series of similar triangles.

6. A plane perpendicular to each of the faces of a diedral angle intersects them in the sides of the plane angle of the diedral angle.

7. If from any point within a diedral angle perpendicular lines are drawn to the faces, the angle between these lines is the supplement of the plane angle of the diedral angle.

8. Find the locus of points equidistant from three given points that are not in one straight line.

9. Prove that a line cannot be perpendicular to each of two intersecting planes.

10. In the first figure of § 307, prove that $OG \perp GF$ by means of the *Theorem of Pythagoras*, using the equations $\overline{OG}^2 + \overline{GE}^2 = \overline{OE}^2$, $\overline{OE}^2 + \overline{EF}^2 = \overline{OF}^2$, $\overline{GF}^2 = \overline{GE}^2 + \overline{EF}^2$. Add these three equations.

11. If the edges of one polyedral angle are perpendicular to the faces of another, the edges of the second are perpendicular to the faces of the first.

12. In any triedral angle the three planes bisecting the three diedral angles intersect in one line, all points of which are equally distant from the three faces.

13. All points within a triedral angle and equally distant from its three faces are in the intersection of the bisecting planes of the diedral angles.

14. The locus of points within a triedral angle and equally distant from its three faces is the line of intersection of the three planes which bisect the diedral angles of the triedral angle.

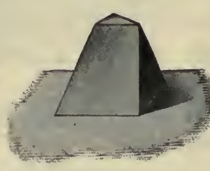
CHAPTER XIII

PRISMS AND CYLINDERS

347. Geometric solids. — A **geometric solid** is a portion of space which is completely inclosed or separated from the rest of space by some kind of surface.

348. Polyedrons. — A **polyedron** is a solid whose bounding surface consists of portions of intersecting planes.

The figure adjoining is a polyedron. It is bounded by portions of six planes.

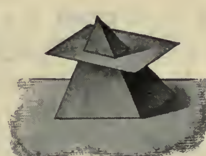
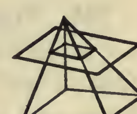



POLYEDRON

The portions of planes which form the bounding surface of a polyedron are called its **faces.** The intersections of the faces are called its **edges.** The intersections of its edges are called its **vertices.** How many edges are there in the polyedron above? How many vertices are there?

A line-segment which joins any two vertices of a polyedron that do not lie in the same face is called a **diagonal** of the polyedron.

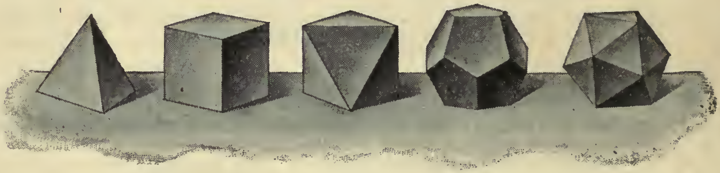
If a plane intersects a polyedron, the polygon formed by the intersections of the plane and the faces is called a **section** of the polyedron.

SECTION OF POLYEDRON

If every section of a polyedron is a convex polygon, it is called a **convex polyedron.** Only convex polyedrons will be considered in this volume.

A polyedron of four faces is called a **tetraedron** ; one of six faces, a **hexaedron** ; one of eight faces, an **octaedron** ; one of twelve faces, a **dodecaedron**; one of twenty faces, an **icosaedron.** The following drawings show models of these polyedrons.



TETRAEDRON HEXAEDRON OCTAEDRON DODECAEDRON ICOSAEDRON

EXERCISES

1. What is the least number of faces that a polyedron can have? Edges? Vertices? What kind of polyedron is it?

2. How many edges has a hexaedron? An octaedron? A dodecaedron? An icosaedron?

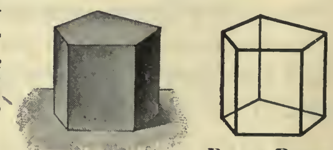
3. How many vertices has a hexaedron? An octaedron? A dodecaedron? An icosaedron?

4. If E is the number of edges, F the number of faces, and V the number of vertices of a polyedron, show that in each of the five polyedrons named above $E + 2 = F + V$. This principle is known as *Euler's Theorem.*

5. Show that in a tetraedron, if S equals the sum of the face angles and V equals the number of vertices, $S = 4(V - 2)$ right angles.

Is this formula true for a hexaedron? For an octaedron? For a dodecaedron? For an icosaedron?

349. Prisms. — A **prism** is a polyedron of which two faces are congruent polygons in parallel planes, and of which the other faces are parallelograms each of which has sides of the polygons as two of its opposite sides.



RIGHT PRISM

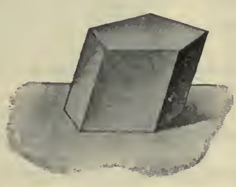
The two congruent polygons in parallel planes are called the **bases**, and the other faces are

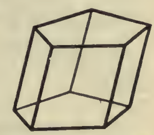
called the **lateral faces.** The intersections of the lateral faces are called the **lateral edges.**

The perpendicular distance between the bases of a prism is called the **altitude** of the prism.

The sum of the areas of the lateral faces of a prism is called the **lateral area.**

The section of a prism which is made by a plane which intersects all of the lateral edges and is perpendicular to them is called a **right section** of the prism.





OBLIQUE PRISM

A prism whose lateral edges are perpendicular to its bases is called a **right prism.**

A prism whose lateral edges are not perpendicular to its bases is called an **oblique prism.**

A prism is called **triangular, quadrangular, hexagonal,** etc., according as its bases are triangles, quadrilaterals, hexagons, etc.

350. Fundamental properties of a prism. — The following important properties of a prism are easily deduced from the definitions above. The student should draw figures and reason out the correctness of each.

(1) *The lateral edges of a prism are parallel and equal.*

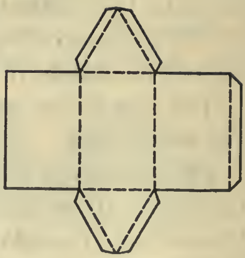
(2) *The altitude of a right prism is equal to a lateral edge of the prism.*

(3) *The lateral faces of a right prism are rectangles.*

(4) *Sections of a prism made by parallel planes cutting all lateral edges are congruent polygons.*

(5) *Right sections of a prism are congruent polygons.*

EXERCISES

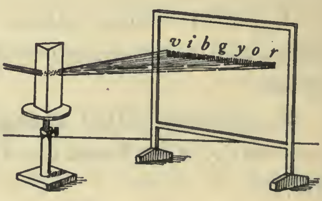


1. On a piece of cardboard, draw a figure similar to the adjoining figure, making each side of each triangle 3 in. and the rectangles 3 in. by 5 in. Cut out the pattern, and, by folding along the dotted lines and pasting, make a model of a triangular prism.

2. What kind of polyedrons are the cells which contain the honey in the comb of the bee? What is the advantage of building the cells in this form?

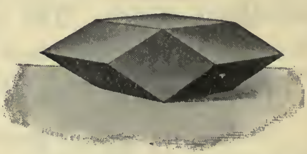
3. In the form of what kind of polyedron are lead pencils sometimes made?

4. A rectangular room or box is what kind of prism?



5. Glass prisms, such as that shown in the drawing, are used in optical instruments for changing the direction of light passing through them. A beam of white light passing through this prism is dispersed or separated into a rainbow-colored band, gradually changing from red at one end, through orange, yellow, etc., to violet at the other end. What kind of prism is it?

Prismatic pieces of glass are used also as pendants or fringes on chandeliers to disperse the light.



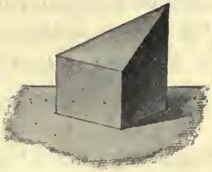
6. The drawing shows a model of a polyedron of which two faces are congruent polygons in parallel planes, and of which the other faces are all parallelograms. Why is it not a prism?

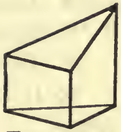
7. A section of a prism made by a plane parallel to a base is congruent to the base.

8. Any section of a prism made by a plane parallel to a lateral edge is a parallelogram.

9. The lateral faces of a right prism are perpendicular to the bases.

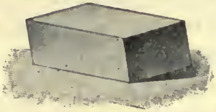
351. Truncated prisms. — The part of a prism contained between a base and a plane which is not parallel to the base is called a **truncated prism.**

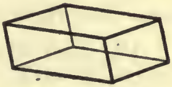




TRUNCATED PRISM

352. Parallelopipeds. — A prism whose bases are parallelograms is called a **parallelopiped.**





PARALLELOPIPED

A parallelopiped whose lateral edges are perpendicular to the bases is a **right parallelopiped.**

A right parallelopiped whose bases are rectangles is a **rectangular parallelopiped.**

A rectangular parallelopiped all of whose faces are squares is a **cube.**

353. Fundamental properties of a parallelopiped. — The following properties of a parallelopiped follow from the definitions above. The student should draw figures and reason out the correctness of each.

(1) *The opposite lateral faces of a parallelopiped are congruent and parallel.*

(2) *Any two opposite faces of a parallelopiped may be taken as the bases.*

(3) *Any right section of a parallelopiped is a parallelogram.*

EXERCISES

1. Find the sum of all of the face angles of any parallelopiped.

2. The diagonals of a rectangular parallelopiped are equal.

3. How, by measuring the diagonals of a piece of house framing, can a carpenter tell when it is truly rectangular?

4. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of three concurrent edges.

5. Find the diagonal of a cube whose edge is 2 in.

6. If the edge of a cube is e, find the length of a diagonal of the cube.

7. The diagonal of a face of a cube is $3\sqrt{2}$. Find the diagonal of the cube.

8. Are the diagonals of a cube perpendicular to each other?

9. A suitcase is 26 in. long, 15 in. high, and 7 in. thick. Can an umbrella which is 32 in. long be packed inside of it?

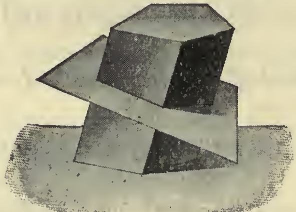
10. Show that the edge, diagonal of a face, and diagonal of a cube are in the ratio of $1 : \sqrt{2} : \sqrt{3}$.

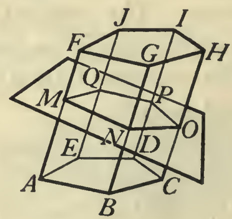
11. The sum of the squares of the four diagonals of any parallelopiped is equal to the sum of the squares of the twelve edges.

12. The diagonals of any parallelopiped all meet at one point, which is the middle point of each.

13. The intersection of the diagonals of any parallelopiped and the intersections of the diagonals of two opposite faces are in a straight line.

354. Theorem. — *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.*





Hypothesis. $MNOP \cdots$ is a right section of a prism, and AF is a lateral edge.

Conclusion. The lateral area $= AF(MN + NO + OP + \text{etc.})$.

Suggestions. Express the area of each lateral face, and form the sum of these areas. Apply § 350 (1). Factor.

355. Corollary. — *The lateral area of a right prism is equal to the product of the altitude and the perimeter of a base.*

The proof is left to the student.

EXERCISES

1. Find the lateral area of a rectangular parallelopiped whose length is 24 in., width 16 in., and height 12 in. Find the total area.

2. Write the formula which expresses the total area of a rectangular parallelopiped whose dimensions are a, b, and c.

3. Find the lateral area of a right hexagonal prism in which each side of a base is 8 in. and the altitude is 20 in.

4. Find the lateral edge of a prism whose lateral area is 714 sq. in. and perimeter of a right section 42 in.

5. A gallon of paint costing $1.50 will cover 300 sq. ft. with one coat. How much will it cost to paint both sides of an 8-foot fence around an athletic field that is 310 ft. wide and 325 ft. long?

6. How many square feet of sheathing are necessary to cover the sides and ends of a box car 34 ft. long, 8 ft. wide, and $7\frac{1}{2}$ ft. high? Add 10 % for waste in cutting and matching.

7. In computing the cost of lathing and plastering a room, the total area of wall space is considered, without any deductions for openings. A room is $9\frac{1}{2}$ ft. high, 15 ft. wide, and 20 ft. long. Laths cost $6 per thousand, and are delivered in bundles of 50 each. A bundle will lath 4 sq. yd. How many bundles are required? (Part of a bundle cannot be bought.) What will they cost?

8. Find the lateral area of a right triangular prism, each side of a base of which is 4 in. and the altitude of which is 9 in. Find also the total area.

9. Find the total area of a right triangular prism whose altitude is 24 in. and whose base is a right triangle in which the sides forming the right angle are 6 in. and 8 in., respectively.

10. Find the lateral area of a right prism whose base is a regular hexagon of which each side is 5 in. and the altitude is 12 in. Find also the total area.

11. The lateral area of a rectangular parallelopiped is 1152 sq. in. and the total area is 1632 sq. in. The height is 18 in. Find the length and breadth.

SUGGESTION. Use a system of equations.

12. The total area of a rectangular parallelopiped is 550 sq. ft. The width is twice as great as the depth, and the length is three times as great as the depth. Find the depth, width, and length.

13. The total area of a cube is 96 sq. in. Find the edge.

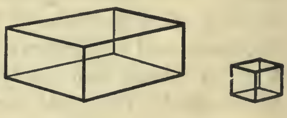
14. If S represents the total area of the surface of a cube, what represents the length of an edge?

15. The lateral area of a right prism whose base is a regular hexagon is $4\,dh\sqrt{3}$, where d is the apothem of the base and h is the altitude of the prism.

16. Write the formula which expresses the total area of the right hexagonal prism in Exercise 15.

17. The three face angles at one vertex of a parallelopiped are each 60°, and the three lateral edges of the triedral angle with that vertex are 2 in., 4 in., 6 in., respectively. Find to two decimal places the area of the entire surface of the solid.

356. Volume of a solid. — The **volume** of a solid is the numerical measure of the solid (See § 115), the unit of measure being a cube whose edge is some linear unit.



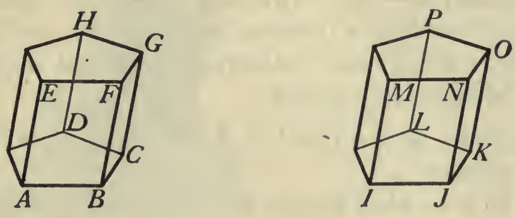
Thus, if a rectangular parallelopiped contains a cube whose edge is 1 unit exactly 24 times, and the cube is taken as the unit of measure, the volume of the parallelopiped is 24.

If an empty rectangular box is 2 in. deep, 4 in. wide, and 5 in. long, how many cubic blocks each 1 in. long could be packed in it? Then what is the volume of the box?

357. Congruent and equal solids. — Two solids which have the same volume are **equal**. It is evident that equal solids need not have the same shape.

Two solids which are alike in all respects, so that they may be made to coincide, are **congruent**.

358. Theorem. — *Two prisms are congruent if three faces including a triedral angle of one are congruent respectively to three faces including a triedral angle of the other, and are similarly placed.*



Hypothesis. Prisms AG and IO have faces BE, BG, BD respectively congruent to faces JM, JO, JL, and similarly placed.

Conclusion. Prism $AG \cong$ prism IO.

Proof. 1. $BE \cong JM$, $BG \cong JO$, $BD \cong JL$. Hyp.

2. $\therefore$ $\angle FBA = \angle NJI$, $\angle CBF = \angle KJN$, $\angle CBA = \angle KJI$. Def. congruence

3. $\therefore$ triedral angles $B\text{–}ACF$ and $J\text{–}IKN$ are equal. § 346

4. Hence by applying prism AG to prism IO, angles $B\text{–}ACF$ and $J\text{–}IKN$ may be made to coincide, face BD coinciding with JL, BE with JM, BG with JO, D falling at L, etc. § 344

5. The lateral edges are parallel. § 350, (1)

6. $\therefore$ DH will fall along LP. § 46

7. $\therefore$ planes CH and KP coincide. § 299

8. Since E, F, G coincide with M, N, O, respectively, planes EG and MO coincide, and hence H and P coincide. § 297

9. Similarly, the other planes of corresponding lateral faces and the other vertices of FH and NP coincide.

10. $\therefore$ prism $AG \cong$ prism IO. § 357

359. Corollary 1. — *Two right prisms having congruent bases and equal altitudes are congruent.*

The proof is left to the student.

360. Corollary 2. — *Two truncated prisms are congruent if three faces including a triedral angle of one are congruent respectively to three faces including a triedral angle of the other, and are similarly placed.*

For the steps in the proof of § 358 apply equally to two truncated prisms.

Draw two truncated prisms satisfying the hypothesis, and think the steps of the proof.

EXERCISES

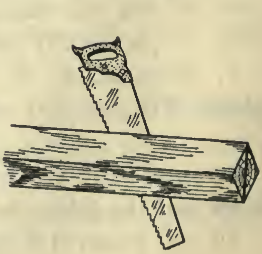
1. Two rectangular parallelopipeds are congruent if the three edges meeting at a vertex of one are equal respectively to the three edges meeting at a vertex of the other.

2. Does the plane determined by two diagonally opposite edges of any parallelopiped divide it into two congruent triangular prisms? Why?

3. Does the plane determined by two diagonally opposite edges of a rectangular parallelopiped divide it into two congruent triangular prisms? Why?

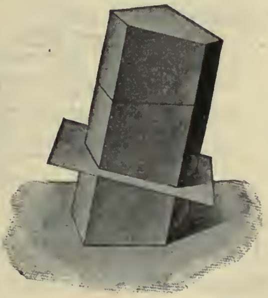
4. Two triangular prisms are congruent if the lateral faces of one are congruent respectively to the lateral faces of the other and are similarly placed.

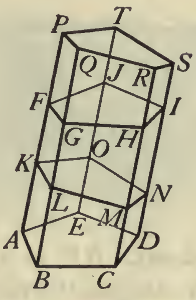
5. If a square wooden beam of which the ends are at right angles to the longitudinal edges is sawed lengthwise along two diagonally opposite edges, the two triangular pieces into which it is cut are congruent.



6. For making a right prism of any desired kind, it is only necessary to have a pattern of a base and a lateral edge. Why?

361. Theorem. — *An oblique prism is equal to a right prism having for a base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism.*





Hypothesis. AI is an oblique prism; KS is a right prism, having for a base $KLM \cdots$, a right section of prism AI, and its altitude KP equal to AF, a lateral edge of prism AI.

Conclusion. Oblique prism $AI =$ right prism KS.

Suggestions. It can be proved that prism $AI =$ prism KS if it is first proved that truncated prism $AN \cong$ truncated prism FS. How?

The latter may be proved if it is first proved that faces AD, AL, AO are congruent respectively to faces FI, FQ, FT. How?

Hence begin by proving faces AD, AL, AO congruent respectively to faces FI, FQ, FT.

Write the proof in full.

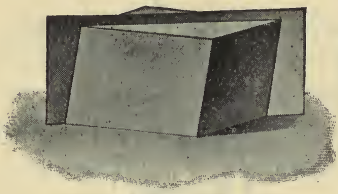
EXERCISES

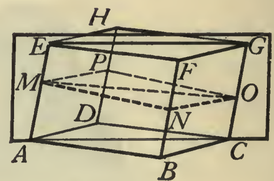
1. Show that the theorem in § 361 may be proved in two ways, by using truncated prisms AN and FS in two ways.

2. Show that an oblique prism may be changed into an equal right prism by cutting it along a right section into two truncated prisms, then interchanging the positions of the latter.

3. Make a model of wood for illustrating the theorem in § 361.

362. Theorem. — *A plane passed through two diagonally opposite edges of any parallelopiped divides it into two equal triangular prisms.*





Hypothesis. BH is any parallelopiped; plane AG passes through diagonally opposite edges AE and CG, forming triangular prisms $ABCEFG$ and $ADCEHG$.

Conclusion. Prism $ABCEFG$ = prism $ADCEHG$.

Proof. 1. Plane AG passes through diagonally opposite edges AE and CG of parallelopiped BH, forming triangular prisms $ABCEFG$ and $ADCEHG$. Hyp.

2. Let $MNOP$ be a right section of BH.

3. Then $MNOP$ is a parallelogram. § 353, (3)

4. $\therefore \triangle MNO \cong \triangle MPO$. § 83

5. Prism $ABCEFG$ = a right prism with $\triangle MNO$ for base and AE for altitude, and prism $ADCEHG$ = a right prism with $\triangle MPO$ for base and AE for altitude. § 361

6. But these right prisms are equal. § 359

7. $\therefore$ prism $ABCEFG$ = prism $ADCEHG$. Ax. I

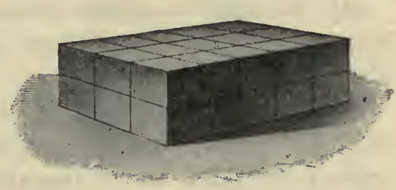
EXERCISES

1. A plane passed through two diagonally opposite edges of a rectangular parallelopiped divides it into two congruent triangular prisms.

2. In the figure of § 362, the plane passed through the diagonally opposite edges AE and CG and the plane passed through the diagonally opposite edges BF and DH divide parallelopiped BH into four equal triangular prisms.

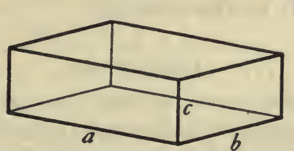
3. A bin in the form of a rectangular parallelopiped that holds 840 bu. of grain is divided into four compartments by two vertical diagonal planes. How many bushels in each compartment?

363. Volume of a rectangular parallelopiped. The lengths of the three edges of a rectangular parallelopiped which meet at any vertex are called the **dimensions** of the parallelopiped.



If the dimensions of a rectangular parallelopiped are 2, 3, and 5 linear units, respectively, it is evident that the parallelopiped may be divided into $2 \times 3 \times 5$, or 30, cubes, each having its edge equal to the same linear unit. Hence if one of these cubes is taken as the unit of measure of the parallelopiped, the *volume* of the parallelopiped equals 30.

In general, if the dimensions of a rectangular parallelopiped are a, b, and c linear units, respectively, the number of unit cubes, or the volume of the parallelopiped, is abc.



This relation also may be proved to be true when the three edges intersecting at any vertex do not have a common unit of measure, or when the dimensions can be expressed only approximately, although the proof is not attempted here. Hence:

The volume of any rectangular parallelopiped is equal to the product of its three dimensions.

364. Corollary 1. — *The volume of any rectangular parallelopiped is equal to the product of its altitude and the area of its base.*

The proof is left to the student.

365. Corollary 2. — *The volumes of two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

For, if M is the volume, b the area of the base, and h the altitude of one parallelopiped, and if N is the volume, b the area of the base, and k the altitude of the other, then by § 364,

$$M = hb \text{ and } N = kb.$$

Hence, by dividing, $\dfrac{M}{N} = \dfrac{hb}{kb}$ or $\dfrac{h}{k}$.

366. Corollary 3. — *The volumes of two rectangular parallelopipeds having equal altitudes are to each other as their bases.*

The proof is left to the student.

367. Corollary 4. — *The volume of a cube is equal to the cube of its edge.*

The proof is left to the student.

EXERCISES

1. Find, in its lowest terms, without multiplication, the ratio of the volumes of two rectangular parallelopipeds whose dimensions are 6 ft., 8 ft., 5 ft., and 9 ft., 12 ft., 10 ft., respectively.

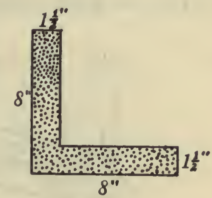
2. A rectangular water tank is 8 ft. 3 in. long and 5 ft. wide. How many cubic feet of water does it contain when the water is 18 in. deep?

3. One cubic foot of iron weighs 450 lb. If an iron bar is $2\frac{1}{2}$ in. thick, 4 in. wide, and 6 ft. 6 in. long, find its weight.

4. A gallon contains 231 cu. in. How many gallons does a tank hold if it is 8 ft. square and 6 ft. high?

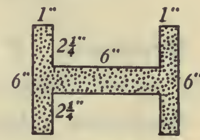
5. In a lot 120 ft. long and 66 ft. wide, a cellar is to be dug for a building. The cellar is to be 44 ft. long, 36 ft. wide, and 7 ft. deep.

The earth removed is to be used for "filling" the surrounding yard What will be the depth of the filling?

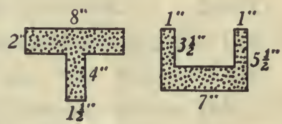


6. In steel construction work, such as the construction of bridges and modern city buildings, the weight of steel is computed and not weighed.

In a certain building contract the specifications require 100 16-foot beams, a right cross section of which is shown in the margin (8″ = 8 in.). Find the weights of the beams and the cost at 5¢ a pound when put in place. Allow 490 lb. per cubic foot.



7. Find the weight of 150 12-foot steel beams, a right cross section of which is shown in the margin.



8. In the construction of a bridge there were T-shaped steel beams 20 ft. long and U-shaped steel beams 16 ft. long, the right cross sections of which are shown in the margin. Compute the weight of a beam of each kind.

9. The total area of a cube is 150 sq. in. Find its volume.

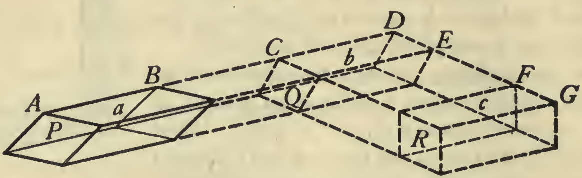
10. The volume of a given cube is v. Find the volume of a cube whose edge is twice as long as the edge of the given cube.

11. The edge of a given cube is e. Find to tenths the edge of a cube containing twice the volume of the given cube.

NOTE. — See § 185. It is discovered in Exercise 11 above that the edge of a cube of which the volume shall be just twice that of a cube of given edge cannot be computed exactly. The corresponding problem of construction, to construct with compasses and unmarked straightedge alone the edge of a cube which shall contain twice the volume of a given cube, is one of the three famous impossible problems of geometry. It dates from the time of the ancient Greeks, and is known as the problem of the *Duplication of the Cube.*

One legend as to the origin of the problem is that the Athenians, who were suffering from a pestilence, consulted the oracle at Delos as to how to stop the plague. Apollo replied through the oracle that in order to stop the pestilence the Athenians must double the size of the altar of the god in Athens. This altar was in the form of a cube. A new altar was constructed with an edge twice as long as the edge of the old one. The pestilence then became worse. The Athenians gave the problem to the mathematicians, who in time proved it impossible of solution.

368. Theorem. — *The volume of any parallelopiped is equal to the product of its altitude and the area of its base.*



Hypothesis. P is any parallelopiped of which the altitude is h and the area of the base is a.

Conclusion. The volume of $P = ha$.

Proof. 1. P is any parallelopiped of which the altitude is h and the area of the base is a. Hyp.

2. Produce the edge AB and all edges parallel to AB. On AB produced, mark off $CD = AB$, and through C and D pass planes perpendicular to AB, forming the right parallelopiped Q.

Produce the edge DE of Q and the edges parallel to DE. On DE produced, mark off $FG = DE$, and through F and G pass planes perpendicular to DE, forming the rectangular parallelopiped R.

3. Then $P = Q$ and $Q = R$. § 361

4. $\therefore P = R$. Ax. I

5. P, Q, and R have a common altitude h. § 323

6. Let the areas of the bases of Q and R be b and c, respectively.

7. Then $a = b$ and $b = c$. § 215

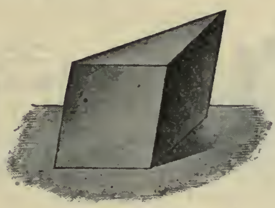
8. $\therefore a = c$. Ax. I

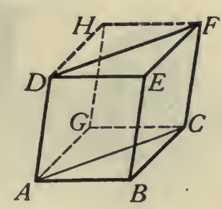
9. The volume of $R = hc$. § 364

10. $\therefore$ the volume of $P = ha$. Ax. XII

Draw a different figure, lettering it differently, and write the proof.

369. Theorem. — *The volume of any triangular prism is equal to the product of its altitude and the area of its base.*





Hypothesis. $ABCDEF$ is any triangular prism of which the altitude is h and the area of the base is a.

Conclusion. The volume of $ABCDEF = ha$.

Proof. 1. The altitude of triangular prism $ABCDEF$ is h and the area of the base is a. Hyp.

2. Construct the parallelopiped BH upon the edges AB, BC, and BE.

3. Then prism $ABCDEF = \frac{1}{2}$ parallelopiped BH. § 362

4. The volume of parallelopiped $BH = h \times ABCG$. § 368

5. And $ABCG = 2a$. § 83

6. $\therefore$ the volume of $ABCDEF = \frac{1}{2} \times h \times 2a = ha$. Ax. XII

EXERCISES

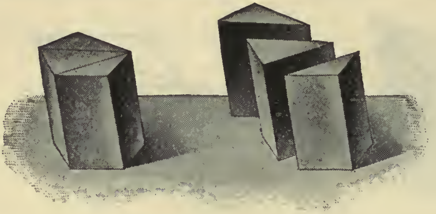
1. Find the volume of a prism whose altitude is 18 in. and whose base is a right triangle with legs 9 in. and 12 in., respectively.

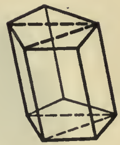
2. Find the volume of a prism whose altitude is 10 in. and whose base is an equilateral triangle with each side 6 in.

3. Find the volume of a triangular prism if the altitude is 14 in. and the sides of the base 8 in., 6 in., 6 in., respectively.

4. Prove that the volume of a triangular prism is equal to one half of the product of any lateral face and the perpendicular to that face from any point in the opposite edge.

370. Theorem. — *The volume of any prism is equal to the product of its altitude and the area of its base.*



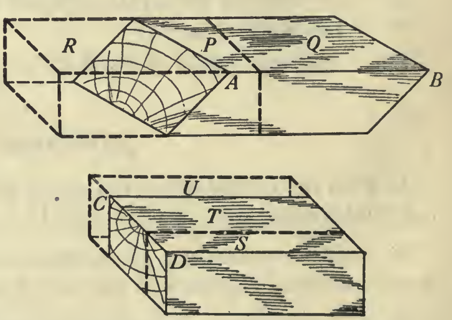


Suggestions. Any prism may be divided into triangular prisms with the same altitude, and with the sum of their bases equal to the base of the given prism, by passing planes through the lateral edges.

Express the volume of each of these triangular prisms, then the sum of the volumes, etc.

Write the proof in full.

EXERCISES



1. Cut out blocks of wood as follows to show that the volume of any parallelopiped is equal to the volume of a rectangular parallelopiped with the same altitude and equal base:

Saw out a block in the form of a parallelopiped of which no face is a rectangle, and let any edge be AB. Saw this into two blocks P and Q by sawing at right angles to AB. Put Q in the new position R. Let CD be an edge at right angles to AB in the new solid. Now saw this into two blocks S and T by sawing at right angles to CD. Put S in the new position U. The resulting solid is a rectangular parallelopiped equal to the given one.

By means of wooden or metal pins, these blocks may be made to stay together in any position in which they may be placed, and hence used as models in the classroom.

2. The volumes of prisms having equal bases and equal altitudes are equal.

3. The volumes of two prisms having equal altitudes are to each other as the areas of the bases.

4. The volumes of two prisms having bases of equal areas are to each other as the altitudes.

5. The volume of any oblique prism equals the product of a lateral edge and the area of a right section.

6. The volume of a right prism whose base is a regular polygon is equal to the product of one half of the apothem of its base and its lateral area.

7. Show by a geometric drawing that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

8. Show by a geometric drawing that $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.

9. The altitude of a prism is 15 in., and its base is a right triangle whose legs are 4 in. and 8 in., respectively. Find its volume.

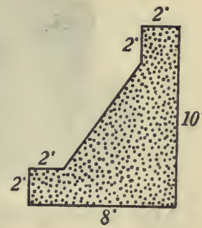
10. The base of a prism is an equilateral triangle each side of which is 5 ft., and its altitude is 9 ft. Find its volume.

11. The base of a prism is a regular hexagon each side of which is 3 in., and its altitude is 18 in. Find its volume.

12. A prism is 20 in. high, and its base is a trapezoid whose parallel sides are 12 in. and 7 in., respectively, and the distance between them 8 in. Find its volume.

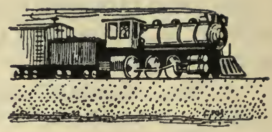
13. A farmer has a corn crib 12 ft. wide and 16 ft. long. The corn in it is piled 10 ft. high along one side, and slopes off to a depth of 6 ft. along the opposite side. Allowing 2 bu. to 5 cu. ft., find how many bushels it contains.

14. Find the cubic contents of a cement retaining wall 120 ft. long, and with a right cross section as shown in the figure. $(2' = 2 \text{ ft.})$



15. The engineer's plan shows a canal trapezoidal in cross section, 9 ft. deep, 12 ft. wide at the bottom, and walls sloping outward at an angle of 45°. The canal is 580 ft. long. The contractor estimates that it will cost him 30¢ a cubic yard to make the excavation. Adding 10% for profit, what should be his bid on the job?

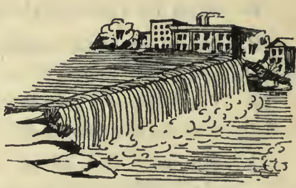
16. Find the number of cubic yards of filling required for a railroad embankment 510 ft. long, if its cross section is trapezoidal, 10 ft. high, 16 ft. wide at the bottom, and 10 ft. wide at the top.



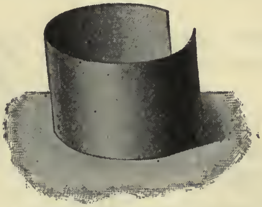
17. The water in an irrigation ditch flows at a speed of 1 ft. a second. The stream runs 3 ft. deep, is 3 ft. wide at the bottom of the ditch and 6 ft. wide at the surface. Find the capacity of the ditch in gallons per second. (Allow $7\frac{1}{2}$ gal. to 1 cu. ft.)

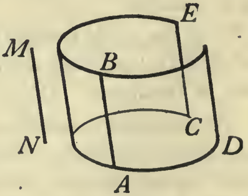
18. Water runs over a V-shaped notch in a dam at a speed of 3 ft. per second. The angle of the notch is 90°, and the water runs to a depth of 18 in. How many gallons of water does the notch deliver per second?

19. The resistance, in pounds, of a masonry dam to sliding because of the pressure of water against it is obtained by multiplying the volume of the masonry, in cubic feet, by 0.75 times the density or weight per cubic foot, of the masonry. A masonry dam whose cross section is a trapezoid 15 ft. high, 3 ft. wide at the top, and 10 ft. wide at the bottom, has a density of 115 lb. to the cubic foot. The dam is 60 ft. long. Find the resistance which it offers to sliding.



371. Cylinders. — A **cylindrical surface** is a curved surface traced by a straight line which moves parallel to a given fixed straight line and continually intersects a guiding curve which is not in the same plane as the fixed line.



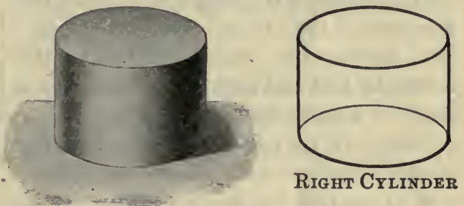


The moving line in any particular one of its positions is called an **element** of the surface.

Thus, if the line AB moves so that it is continually parallel to the line MN and constantly intersects the curve CAD, it traces the cylindrical surface ED. In any particular position, AB is an element of the surface ED.

A **cylinder** is a solid whose bounding surface consists of a portion of a cylindrical surface, called the **lateral surface**, and portions of two parallel planes, called the **bases**, which intersect all elements of the cylindrical surface.

The **altitude** of a cylinder is the perpendicular distance between the bases.



RIGHT CYLINDER

A **section** of a cylinder is the figure formed by the intersection of the cylinder and a plane. A **right section** of a cylinder is a section made by a plane which intersects all of the elements of the cylindrical surface and is perpendicular to them.

A **right cylinder** is a cylinder whose elements are perpendicular to the bases.

OBLIQUE CYLINDER

An **oblique cylinder** is a cylinder whose elements are not perpendicular to the bases.

A **circular cylinder** is a cylinder whose bases are circles.

A **right circular cylinder** is a right cylinder whose bases are circles.

372. Fundamental properties of a cylinder. — The following important properties of a cylinder are easily deduced from

the definitions given above, and should be established by the student:

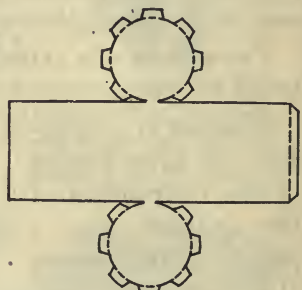
(1) *The elements of a cylinder are parallel.*

(2) *The elements of a cylinder are equal.*

(3) *The altitude of a right cylinder is equal to an element.*

EXERCISES

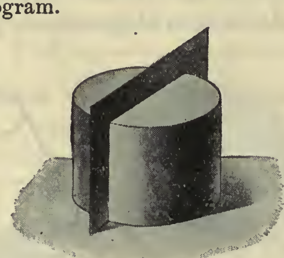
1. On a piece of cardboard draw a figure similar to the adjoining figure, making the length of the rectangle $3\frac{1}{7}$ times the diameter of the circles. Cut out the pattern, leaving the circles attached to the rectangle, fold and paste, and thus make a model of a right circular cylinder.

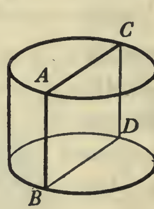


NOTE. — If the school has a lathe, turn out *wooden* cylinders on the lathe for use in class.

2. A straight line drawn through a point in a cylindrical surface parallel to a given element is itself an element.

3. Every section of a cylinder made by a plane containing an element is a parallelogram.



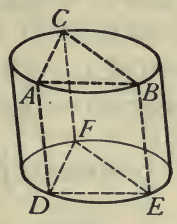


SUGGESTIONS. — Let $ABDC$ be the section containing the given element AB and intersecting the cylindrical surface again in some kind of line CD. Prove that CD is an element. First, draw the element through C.

4. Every section of a right cylinder made by a plane containing an element is a rectangle.

5. Every section of a cylinder made by a plane parallel to a given element is a parallelogram.

373. Theorem. — *The bases of any cylinder are congruent.*



Hypothesis. ABC and DEF are the bases of a cylinder.

Conclusion. $ABC \cong DEF$.

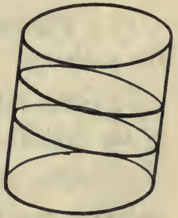
Suggestions. Let A and B be any two points in the perimeter of ABC, and C any third point of the perimeter. Draw the elements AD, BE, CF. Draw AC, BC, AB, DF, EF, DE.

Prove that $\triangle ABC \cong \triangle DEF$.

Then show that base ABC may be superposed upon base DEF so that A and B coincide with D and E, respectively, and that when this is done, C falls upon F, and hence all points in the perimeter of ABC fall at the same time upon corresponding points in the perimeter of DEF, etc.

EXERCISES

1. The sections of a cylinder made by two parallel planes cutting all of the elements are congruent.

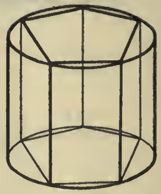


2. Every section of a cylinder made by a plane parallel to a base is congruent to the base.

3. Every section of a circular cylinder made by a plane parallel to a base is a circle.

4. The straight line joining the centers of the bases of a circular cylinder passes through the centers of all sections made by planes parallel to the bases.

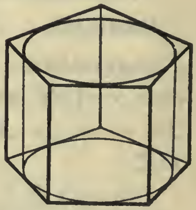
374. Inscribed and circumscribed prisms and cylinders. — A prism is said to be **inscribed** in a circular cylinder when the bases of the prism are inscribed in the bases of the cylinder and the lateral edges of the prism are elements of the cylinder.



INSCRIBED PRISM

The cylinder is said to be **circumscribed** about the prism.

A prism is said to be **circumscribed** about a circular cylinder when the bases of the prism are circumscribed about the bases of the cylinder and the lateral edges are parallel to the elements of the cylinder.



CIRCUMSCRIBED PRISM

The cylinder is said to be **inscribed** in the prism.

It is evident that each lateral face of a circumscribed prism of a cylinder contains one element and no other line nor point of the cylinder. Such a plane is said to be **tangent** to the cylinder.

375. Limits. — The following principles of limits may be assumed:

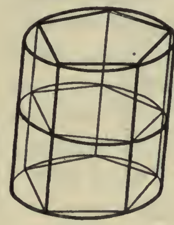
If a prism whose bases are regular polygons is inscribed in, or circumscribed about, a circular cylinder, and if the number of lateral faces is indefinitely increased:

(1) *The perimeter of a right section of the prism approaches the perimeter of a right section of the cylinder as a limit.*

(2) *The lateral area of the prism approaches the lateral area of the cylinder as a limit.*

(3) *The volume of the prism approaches the volume of the cylinder as a limit.*

376. Theorem. — *The lateral area of a circular cylinder is equal to the product of an element and the perimeter of a right section of the cylinder.*



Hypothesis. $S =$ lateral area of a circular cylinder, $P =$ perimeter of a right section, and $E =$ an element.

Conclusion. $S = EP$.

Proof. 1. $S =$ lat. area, $P =$ perim. rt. sec., $E =$ element. Hyp.

2. Inscribe a prism whose base is a regular polygon. Let $s =$ lat. area and $p =$ perim. of right section of prism.

3. Then the lateral edge of the prism $= E$. § 374

4. $\therefore s = Ep$. § 354

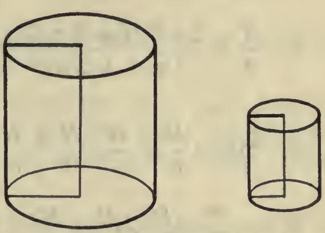
5. But $s \doteq S$ and $p \doteq P$. § 375

6. Since $p \doteq P$, then $Ep \doteq EP$. § 275, (2)

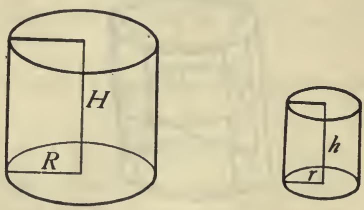
7. $\therefore S = EP$. § 275, (1)

377. Corollary. — *The lateral area of a right circular cylinder is equal to the product of the altitude and the circumference of its base.*

378. Similar cylinders. — Since a right circular cylinder may be formed by revolving a rectangle about a side as axis, it is called a **cylinder of revolution.** Two such cylinders formed by revolving similar rectangles about homologous sides as axes are called **similar cylinders of revolution.**



379. Theorem. — *The lateral or total areas of two similar cylinders of revolution are to each other as the squares of the radii of their bases or as the squares of their altitudes.*



Hypothesis. Of two similar cylinders of revolution, the lateral areas are S and s, total areas T and t, altitudes H and h, and radii of bases R and r, respectively.

Conclusion. $\frac{S}{s}=\frac{T}{t}=\frac{R^2}{r^2}=\frac{H^2}{h^2}.$

Proof. 1. Lateral areas are S and s, total areas T and t, altitudes H and h, radii of bases R and r, respectively. Hyp.

2. $\therefore \frac{S}{s}=\frac{2\pi RH}{2\pi rh}=\frac{RH}{rh}=\frac{R}{r}\times\frac{H}{h}.$ § 377

3. But $\frac{H}{h}=\frac{R}{r}.$ § 127

4. $\therefore \frac{S}{s}=\frac{R}{r}\times\frac{R}{r}=\frac{R^2}{r^2}=\frac{H^2}{h^2}.$ Ax. XII

5. $\frac{T}{t}=\frac{2\pi RH+2\pi R^2}{2\pi rh+2\pi r^2}=\frac{R(H+R)}{r(h+r)}=\frac{R}{r}\times\frac{H+R}{h+r}.$ § 377, § 281

6. Since $\frac{H}{h}=\frac{R}{r}$, $\frac{H+R}{h+r}=\frac{R}{r}=\frac{H}{h}.$ § 118, (8)

7. $\therefore \frac{T}{t}=\frac{R}{r}\times\frac{R}{r}=\frac{R^2}{r^2}=\frac{H^2}{h^2}.$ Ax. XII

EXERCISES

1. Prove the theorem in § 376 by circumscribing about the cylinder a prism whose base is a regular polygon.

2. The total area of a right circular cylinder is equal to the sum of the altitude and radius of the base, multiplied by the perimeter of the base.

3. The lateral areas of the cylinders formed by revolving a rectangle about each of two adjacent sides are equal.

4. If the total area of the surface of a right circular cylinder is T, and the radius of the base R, what is the altitude?

5. If the lateral area of the surface of a right circular cylinder is S, and the altitude H, find the radius of the base.

6. Find the lateral area and also the total area of a right circular cylinder whose altitude is 6 in. and diameter of base 4 in.

7. How many square inches of tin are required for making an open cylindrical pail 10 in. in diameter and 12 in. deep?

8. The altitudes of two similar cylinders of revolution are 3 in. and 4 in., respectively, and the lateral area of the smaller is 45 sq. in. Find the lateral area of the larger.

9. The total areas of two similar cylinders of revolution are 64 sq. in. and 144 sq. in., respectively, and the diameter of the smaller is 6 in. Find the diameter of the larger.

10. The total area of a right circular cylinder is 192 sq. in., and the altitude is 8 in. Find the diameter.

11. In punching round holes through metal plates, the pressure exerted by the punch, in pounds, in the ordinary run of work, must be 60,000 times the area, in square inches, of the cylindrical surface of the hole made. Find the pressure required to punch a hole $\frac{3}{4}$ in. in diameter through a steel plate $\frac{5}{8}$ in. thick.

12. Find the pressure required to punch a hole $\frac{1}{2}$ in. in diameter through a piece of boiler plate $\frac{7}{16}$ in. thick.

13. In a steam engine, 72 flues, or cylindrical pipes, each 2 in. in outside diameter and 14 ft. long, convey the heat from the fire box through the water. How much heating surface do they apply to the water?

14. A condenser contains 800 tubes, each $\frac{3}{4}$ in. in diameter and $6\frac{1}{2}$ ft. long. Find the total area of their cooling surface.

380. Theorem. — *The volume of any circular cylinder is equal to the product of the altitude and the area of the base.*

Hypothesis. $V =$ volume of circular cylinder, $A =$ area of base, $H =$ altitude.

Conclusion. $V = HA.$

Suggestions. Circumscribe about the cylinder a prism whose base is a regular polygon. Let $v =$ volume of prism, $a =$ area of its base. Then proceed as in the proof of § 376.

Write the proof in full.

381. Corollary. — *If R denotes the radius of the base and H the altitude of a circular cylinder, then $V = \pi R^2 H$.*

The proof is left to the student.

382. Theorem. — *The volumes of two similar cylinders of revolution are to each other as the cubes of the radii of their bases, or as the cubes of their altitudes.*

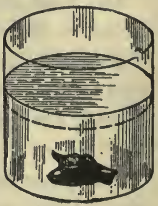
The proof is left to the student. Proceed as in § 379.

EXERCISES

1. Prove the theorem in § 380 by inscribing a prism whose base is a regular polygon.

2. The volume of a right circular cylinder is equal to the product of the lateral area and one half of the radius of the base.

3. The diameter of a well is 8 ft., and the water is 9 ft. deep. Allowing $7\frac{1}{2}$ gallons to 1 cubic foot, find the number of gallons of water in the well.



4. An easy method that may be used for finding the volume of an irregular solid that may be put into water is to immerse it in water in a cylindrical vessel, note the diameter of the vessel, and the depth of the water before and after the solid is put in, and then compute the volume of the water displaced by the solid.

A stone is immersed in a cylindrical jar of water. The diameter of the jar is 8 in., the depths of the water before and after the stone is put in are $5\frac{1}{2}$ in. and $6\frac{3}{4}$ in., respectively. Find the volume of the stone.

5. A jagged piece of iron is immersed in a cylindrical jar of water whose diameter is 10 in. The depths of the water before and after the iron is put in are 7 in. and $9\frac{1}{2}$ in., respectively. Find the volume of the iron.

6. A farmer builds a silo 18 ft. in diameter and 32 ft. high. If a cubic foot of silage weighs 25 lb., how many tons of silage does the silo hold?

7. A sheet-metal worker wishes to construct a cylindrical tank of galvanized iron that will hold $63\frac{1}{2}$ gal. and that will fit into a given space 51 in. high. Find the diameter.

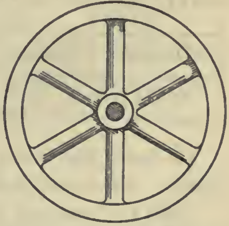
8. A wash boiler 12 in. deep, 10 in. wide, and 20 in. long, has round ends, *i.e.* each end is a half cylinder. How many gallons does it hold?

9. In practical work men use many rules of thumb. It is desirable often to test the accuracy of these rules. For example:

To obtain the weight of round iron, multiply the square of the diameter in inches by the length in feet, and by 2.63. This gives the weight in pounds approximately. Test the accuracy of this rule when applied to a 2-inch iron rod 12 ft. long, allowing 480 lb. to 1 cu. ft.

10. It often is found necessary in machine-shop work to compute the weight of the rim of a flywheel.

The rim of a flywheel is 6 in. wide, the outer diameter 4 ft. 4 in., and the inner diameter 4 ft. Find its weight. Allow 480 lb. to 1 cu. ft.



11. Find the weight of a hollow iron column 12 ft. long, 10 in. in outside diameter, and 1 in. thick, allowing 480 lb. to 1 cu. ft.

12. For irrigating a 5-acre field, water is delivered through an 8-inch pipe at a speed of 2 ft. a second. How long will it take to deliver 2 in. of water over the entire field?

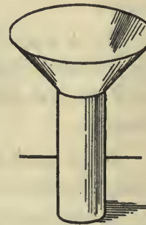
13. In the manufacture of lard, a cylinder for cooling it is $3\frac{1}{2}$ ft. in diameter and 8 ft. long. As the cylinder revolves, the hot lard adheres

to the surface to a depth of $\frac{1}{8}$ in. and is taken off. How many pounds of lard are cooled at each revolution? How many in an hour, if the cylinder makes 5 revolutions a minute? Count $56\frac{1}{4}$ lb. to 1 cu. ft.

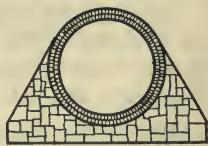
14. A lard tank 4 ft. in diameter and 8 ft. deep has a jacket around it, on the bottom and side, 4 in. from the surface. How many gallons of water will the space between the tank and jacket hold?

15. A cylinder of a steam pump, used for pumping city water, is 2 ft. in diameter and 3 ft. long. It is filled and emptied twice at each revolution of the piston. Find the number of gallons delivered by the pump in a minute if the piston makes 24 revolutions a minute.

16. The figure represents a view of a rain gauge, used for measuring the amount of rainfall. The opening at the top is 12 in. in diameter, and the cylindrical stem is 4 in. in diameter. Suppose that in a rain the stem is filled to a depth of 4 in. What is the precipitation, *i.e.* what is the depth of the rainfall on the level ground?



17. The drawing shows the plan of a cross section of a culvert 40 ft. long. The inner diameter of the brickwork is 8 ft., and the outer diameter 9 ft. 6 in. Allowing 10% for space taken up by mortar, find how many thousand bricks must be ordered for use in its construction.



18. The Spaulding rule for computing the number of board feet in 16-foot logs is as follows:

Diameter in inches	12	16	20	24	28
Board feet	77	161	276	412	569

(A *board foot* is the quantity of timber in a board 1 ft. square and 1 in. thick. It is equivalent to $\frac{1}{12}$ of a cubic foot.)

Find the amount, in cubic feet, that is wasted in sawing up a 20-inch log, if figured by this rule. Find the per cent wasted.

19. The Doyle rule, which is the rule most generally used throughout this country for computing the number of board feet that a given log will make when sawed, is as follows:

"Deduct 4 in. from the diameter of the log as an allowance for slab; square one quarter of the remainder; and multiply the result by the length of the log in feet."

According to this rule, find the amount, in cubic feet, that is wasted in sawing a 12-foot log that is 25 in. in diameter. Find the per cent wasted.

20. A log 10 ft. long and 4 ft. in diameter is decayed at the center. The diameter of the decayed part is 10 in. Find the number of cubic feet of good timber.

NOTE. — The amount left or wasted that is required in Exercises 20, 21, and 22 does not refer to the board feet of lumber, but merely to that part of the volume of the log left or wasted.

21. A log 3 ft. in diameter and 12 ft. long has a defect on the surface, due to sun scald, which causes a waste of a slab. The defect extends over one fourth of the surface. Find the number of cubic feet in the slab wasted.

SUGGESTION. — The area of a right section is the difference between a sector of a circle and a triangle.

22. A log 4 ft. in diameter and 16 ft. long has a defect due to decay which causes a waste of a part of the log whose right section is a sector of a circle of which the angle at the center is 60°. Find the number of cubic feet in the part wasted.

23. The "Inscribed Square" rule, a rule of thumb used in lumbering, gives the cubic contents of square pieces of timber that can be cut from cylindrical logs. The width of the square piece is obtained by multiplying the diameter of the log by 17, and dividing the result by 24. Is this accurate?

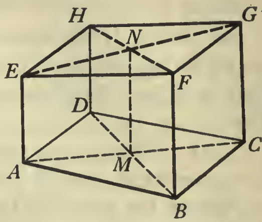
MISCELLANEOUS EXERCISES

1. A section of a tetraedron made by a plane parallel to any face is a triangle.

2. The section of a tetraedron made by a plane which is determined by the middle points of any three edges that do not all meet at one vertex is a parallelogram.

3. The line-segments which join the middle points of opposite edges of any tetraedron intersect at one point.

4. Every pair of lateral edges of a prism determines a plane which is parallel to every other lateral edge of the prism.

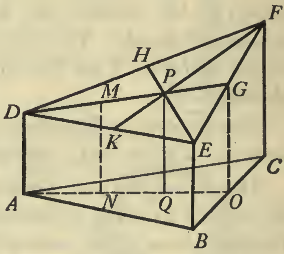


5. The upper base of a truncated parallelopiped is a parallelogram.

6. The sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

SUGGESTION. — Compare $AE + CG$ with MN, and compare $BF + DH$ with MN.

7. The perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base is equal to one third of the sum of the three lateral edges.



SUGGESTION. — Let M be the middle point of DP. Draw $MN \perp ABC$. Now express PQ in terms of MN and GO; also express MN in terms of DA and PQ. Eliminate MN.

8. A man contracted to excavate a cellar at 80¢ per cubic yard. The location was on sloping ground, so that the depth of the cellar at the upper side was 8 ft. and at the lower side 4 ft. The length of the cellar from the front or lower side to the back or higher side was 32 ft. and the width was 28 ft. What price was the contractor paid for the work?

9. In the manufacture of many cylindrical articles from sheet metal, such as can lids, shoe-polish boxes, etc., circular blanks are first cut from flat sheet metal. These are then pressed into the required shape in a die. In some large factories one man is kept busy computing the sizes of blanks. The computation is based upon the assumption that the total area of the finished article equals the area of the circular blank. The modification necessary, due to the stretching of the metal, is afterward found by trial.

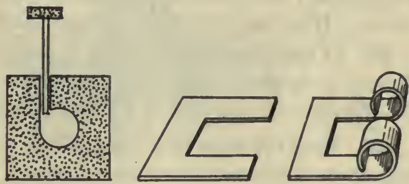
The lid of a 3-pound lard bucket is $5\frac{1}{4}$ in. in diameter, and has a $\frac{1}{2}$-inch flange. Find the diameter of the blank.

10. A small sample vaseline box is $\frac{1}{2}$ in. deep and $1\frac{1}{2}$ in. in diameter. Find the diameter of the blank from which it is made.

11. A Shinola shoe-polish box lid is $2\frac{5}{8}$ in. in diameter, and has a $\frac{3}{8}$-inch flange. Find the diameter of the blank from which it is made.

12. A Rumford baking-powder can lid is 3.1 in. in diameter, and has a $\frac{1}{2}$-inch flange. Find the diameter of the blank from which it is made.

13. In the manufacture of door hinges, the blank shown in the middle figure is first cut from flat sheet metal, and then bent into the form shown at the right by forcing it into a die. The die consists of a heavy piece of metal into which is cut a vertical slot just wide enough to admit the blank edgewise. This vertical slot terminates in a cylindrical hole. By great pressure against its edge, the blank is forced into the slot, and follows around the wall of the hole, thus being bent into the required form.



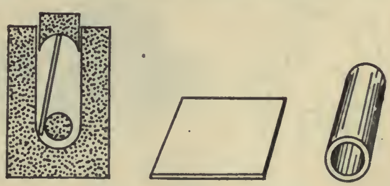
The size of blank that must be cut is first computed theoretically, and because of the stretching on one side of the metal and compression on the other, modifications to be made in the dimensions are found by trial.

The flat part of a steel door hinge is to be 4 in. long and $1\frac{3}{4}$ in. wide, and the outside diameter of the cylindrical part $\frac{1}{2}$ in. Find the size that the blank must be made.

14. In a small brass hinge used in the manufacture of furniture, the flat part is to be $1\frac{1}{4}$ in. by $\frac{3}{4}$ in., and the outside diameter of the cylindrical part $\frac{3}{16}$ in. Find the dimensions of the blank that must be cut.

15. The flat part of a small steel hinge is 1 in. by 4 in., and the outside diameter of the cylindrical part .2 in. Find the dimensions of the blank.

16. Metal tubes are made by a process similar to that used in the manufacture of hinges, except that the die is slightly different, as shown in the drawing. Find the dimensions of the blanks for making brass tubes used as curtain poles that are 22 in. long and $\frac{3}{8}$ in. in diameter.



How many square feet of sheet brass will it take to make 1000?

17. In the manufacture of brass chandeliers, a tube used is 1 in. in diameter and 3 ft. long. How much sheet brass is required to make 1000 of these tubes?

18. How high must a tomato can that is to hold a quart be made, if its diameter is 4 in.? Allow 231 cu. in. to a gallon.

19. Find the length of a wire $\frac{1}{16}$ in. in diameter that can be drawn from a cubic foot of brass.

20. A boiler of an engine 4 ft. in diameter and 16 ft. long is traversed by 60 pipes, each 3 in. in diameter, which convey the heat through the water. How many gallons of water does the boiler hold?

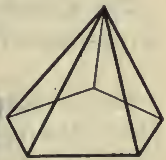
21. If the length of a tube is l, and the outer and inner diameters are D and d, respectively, prove that the volume equals $\frac{1}{4}\pi l(D^2 - d^2)$. If the thickness of the tube is t, show that the volume equals $\frac{1}{2}\pi lt(D + d)$.

22. Find the edge of a cube whose volume and area of the entire surface contain the same number of units.

CHAPTER XIV

PYRAMIDS AND CONES

383. Pyramids. — A **pyramid** is a polyhedron of which one face, called the **base**, is a polygon of any number of sides, and of which the other faces are triangles having a common vertex.

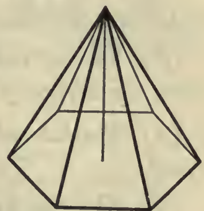


PYRAMID

The triangles are called the **lateral faces**, and their common vertex the **vertex** of the pyramid. The edges which meet at the vertex of the pyramid are called the **lateral edges**.

The sum of the areas of the lateral faces is called the **lateral area.**

The perpendicular distance from the vertex to the plane of the base is called the **altitude** of the pyramid.

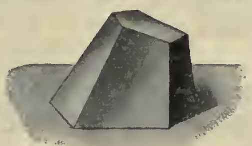


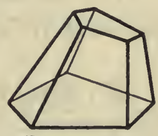
REGULAR PYRAMID

A pyramid is called **triangular, quadrangular, hexagonal**, etc., according as its base is a triangle, quadrilateral, hexagon, etc.

A **regular pyramid** is a pyramid whose base is a regular polygon and whose vertex lies in the perpendicular to the base at its center.

384. A truncated pyramid. — A **truncated pyramid** is the part of a pyramid included between its base and a plane cutting all of the lateral edges.



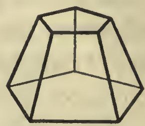


TRUNCATED PYRAMID

The base of the pyramid and the section of the cutting plane are called the **bases** of the truncated pyramid.

385. A frustum of a pyramid. — A **frustum of a pyramid** is a truncated pyramid of which the bases are parallel.

The **altitude** of a frustum of a pyramid is the perpendicular distance between the bases.



FRUSTUM OF PYRAMID

386. Fundamental properties of pyramids. — The following important properties of pyramids are easily deduced from the definitions given above.

The student should draw figures, and reason out the correctness of each.

(1) *The lateral edges of a regular pyramid are equal.*

(2) *The lateral faces of a regular pyramid are congruent isosceles triangles.*

(3) *The altitudes of the faces of a regular pyramid drawn from the vertex of the pyramid are equal.*

(4) *The lateral edges of a frustum of a regular pyramid are equal.*

(5) *The lateral faces of a frustum of a regular pyramid are congruent trapezoids.*

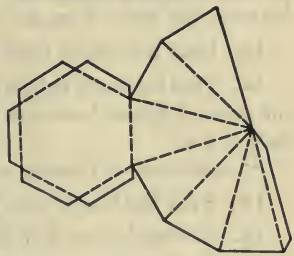
(6) *The altitudes of the faces of a frustum of a regular pyramid are equal.*

387. Slant height. — The altitude of a lateral face of a regular pyramid, drawn from the vertex of the pyramid, is called the **slant height**.

The altitude of a lateral face of a frustum of a regular pyramid is called the **slant height** of the frustum.

EXERCISES

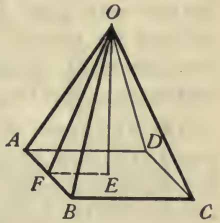
1. Draw on cardboard a figure similar to the adjoining figure, making each side of the hexagon 2 in. Cut out the pattern, and by folding along the dotted lines and pasting, make a model of a regular hexagonal pyramid.



2. If E, F, G, and H are the middle points of the edges AB, AD, CD, and BC, respectively, of the triangular pyramid $A\text{–}BCD$, then $EFGH$ is a parallelogram.

Suggestion. — What is the relation of EH and FG to AC? Could it be proved by using a different pair of lines?

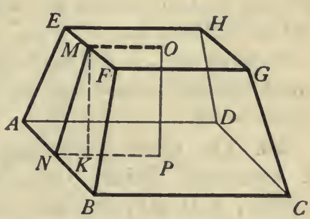
3. Find the slant height of a regular quadrangular pyramid whose altitude is 4 in. and each side of whose base is 6 in.



Suggestions. — Let OE be the altitude of the regular quadrangular pyramid $O\text{–}ABCD$, and let OF be the slant height. Draw FE. Find FE. Then find OF.

4. Find the lateral edge of the pyramid in Ex. 3.

5. Find the slant height of a regular hexagonal pyramid whose altitude is 12 in. and each side of whose base is 4 in.



6. Find the lateral edge of the pyramid in Ex. 5.

7. The altitude of a frustum of a regular quadrangular pyramid is 8 in. and the sides of the bases are 12 in. and 16 in., respectively. Find the slant height.

Suggestions. — Find MO and NP. Then find NK and MK. Then find MN.

8. Find the lateral edge of the frustum in Ex. 7.

9. Find the slant height of a regular triangular pyramid each side of whose base is 3 in. and whose altitude is 4 in.

SUGGESTION. — The apothem of the base is a leg of a right triangle whose acute angles are 30° and 60° respectively. What is the relation between the sides of such a triangle?

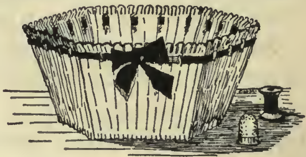
10. Find the lateral edge of the pyramid in Ex. 9.

11. Find the slant height of a frustum of a regular triangular pyramid the sides of whose bases are 9 in. and 5 in., respectively, and whose altitude is 4 in.

SUGGESTION. — Proceed as in Exercise 7.

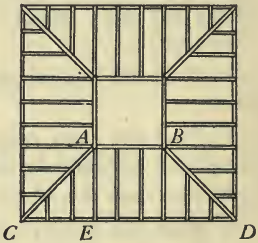
12. Find the lateral edge of the frustum in Ex. 11.

13. A work basket is in the form of a frustum of a regular hexagonal pyramid whose slant height is $4\frac{1}{2}$ in. and the sides of whose top and bottom bases are $5\frac{1}{2}$ in. and 4 in., respectively. The sides and bottom are covered inside and outside with silk. If one half extra is allowed for fullness in shirring the silk, how much silk is required for the basket? If the silk is 27 in. wide, find to an eighth of a yard how much of a yard is required.



14. The figure shows the plan of a square roof in the form of a frustum of a pyramid, the upper base being a flat deck. CD is 18 ft., AB is 6 ft., and the height of the roof, or altitude of the frustum, is 8 ft. Find the lengths that the rafters AC and AE must be cut in building it.

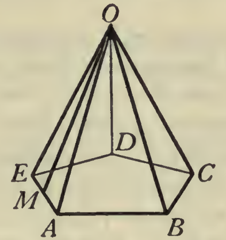
SUGGESTION. — AE is the hypotenuse of a right triangle whose legs are 6 ft. and 8 ft. respectively.



15. The figure shows the plan of a roof of a hexagonal tower. OA, OB, etc., are hip rafters; CD, EF, etc., are jack rafters. If the slope of the roof is 45°, and the length of AB is 8 ft., find the length that each rafter must be cut in building it.

16. Prove that the perimeter of the mid-section of a frustum of any pyramid, made by a plane parallel to the bases, is equal to one half of the sum of the perimeters of the bases.

388. Theorem. — *The lateral area of any regular pyramid is equal to one half of the product of the slant height and the perimeter of the base.*

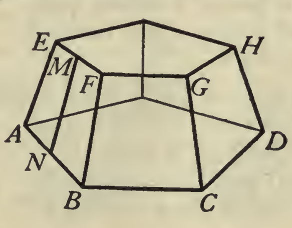


Hypothesis. $O\text{–}ABC \cdots$ is a regular pyramid; $S =$ its lateral area; $l =$ its slant height; $p =$ the perimeter of its base.

Conclusion. $S = \frac{1}{2}\, lp.$

Proof. The proof is left to the student. Write the proof in full.

389. Theorem. — *The lateral area of a frustum of a regular pyramid is equal to one half of the product of the slant height and the sum of the perimeters of the bases.*



Hypothesis. AH is a frustum of a regular pyramid; $S =$ its lateral area; $l =$ its slant height; $p =$ perimeter of one base and $q =$ perimeter of the other base.

Conclusion. $S = \frac{1}{2}\, l(p + q).$

Proof. The proof is left to the student. Write the proof in full.

EXERCISES

1. The lateral area of a regular pyramid is equal to the product of the slant height and the perimeter of a mid-section made by a plane parallel to the base.

2. The lateral area of a pyramid is greater than the area of the base.

SUGGESTION. — Draw line-segments from the foot of the altitude to all of the vertices of the base, dividing the base into triangles which may be compared to the corresponding lateral faces.

3. Prove the theorem in § 388 as a corollary of the theorem in § 389, by reducing one base of the frustum to a point.

4. The slant height of a regular pyramid is 24 ft., and the base is a triangle each side of which is 10 ft. Find the lateral area. The total area.

5. The base of a regular pyramid is a hexagon each side of which is 16 in., and the slant height is 20 in. Find the lateral area. The total area.

6. A tent is made of canvas stretched tightly over six poles, which are tied together at the top, and has the form of a regular pyramid. The distance between the feet of each two adjoining poles is $4\frac{1}{2}$ ft., and the slant height of the tent is 14 ft. How many square yards of canvas are there in the cover?

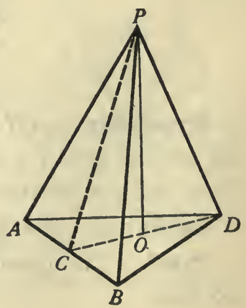
7. Find the lateral area of a regular quadrangular pyramid each side of whose base is 6 ft., and whose altitude is 8 ft.

8. The Great Pyramid of Egypt, when completed, was 481 ft. high, and each side of its square base was 764 ft. long. How many acres were there in its surface?

9. Find the lateral area of a regular hexagonal pyramid each side of whose base is 4 ft., and whose altitude is 10 ft.

10. Each side of the base of a regular triangular pyramid is 4 ft., and its altitude is 6 ft. Find the lateral area.

SUGGESTION. — Let PAB be one face; let O be the center of the base; and let $OC \perp AB$. First compute OC; then PC. See § 114.



11. Find the lateral area of a triangular pyramid whose lateral faces are all equilateral triangles and whose altitude is 4 ft.

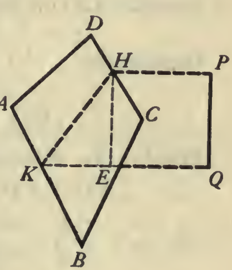
SUGGESTION. — The apothem of the base equals one third of the slant height.

12. Find the lateral area of a frustum of a regular triangular pyramid the sides of whose bases are 42 in. and 72 in., respectively, and whose slant height is 36 in.

13. The pedestal of a marble column is in the form of a frustum of a pyramid whose bases are regular octagons with sides 3 ft. and 2 ft. 8 in. respectively, and whose slant height is 14 in. How much surface must be polished?

14. A marble monument consists of a frustum of a square pyramid whose bases are 2 ft. and 1 ft. 10 in. square, respectively, and whose slant height is 5 ft., surmounted by a pyramid whose slant height is 2 ft. What is the amount of polished surface?

15. Find the lateral area of a frustum of a regular quadrangular pyramid the sides of whose bases are 22 in. and 12 in., respectively, and whose altitude is 16 in.



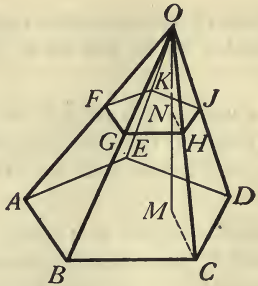
SUGGESTIONS. — Let $ABCD$ be a lateral face; let P and Q be the centers of the bases; let $PH \perp CD$ and $QK \perp AB$; and let $HE \perp QK$. First compute KE, then HK.

16. Find the lateral area of a frustum of a regular triangular pyramid the sides of whose bases are 16 ft. and 12 ft., respectively, and whose altitude is 12 ft.

17. Find the lateral area of the frustum of a regular hexagonal pyramid the sides of whose bases are 9 ft. and 5 ft., respectively, and whose altitude is 4 ft.

18. A church spire which is in the form of a regular hexagonal pyramid is covered with slate. Each side of the base is 6 ft., and a lateral edge is 38 ft. In figuring on the contract for building the spire, the contractor must estimate the amount of slate required to cover it. How many squares (a square is 100 sq. ft.) of slate are required?

390. Theorem. — *If a pyramid is cut by a plane parallel to the base: (1) the lateral edges and the altitude are divided proportionally; (2) the section is a polygon similar to the base.*



Hypothesis. In pyramid $O\text{–}ABC\cdots$, plane of section $FGH\cdots$ is parallel to the base and meets altitude OM at N.

Conclusion. (1) $\frac{OF}{OA}=\frac{OG}{OB}=\cdots=\frac{ON}{OM}$;

(2) $FGH\cdots\sim ABC\cdots$.

Proof. 1. Plane $FH \parallel$ plane AC. Hyp.

2. $\therefore FG \parallel AB$, $GH \parallel BC$, etc., and $NH \parallel MC$. § 315

3. $\therefore \frac{OF}{OA}=\frac{OG}{OB}=\cdots=\frac{ON}{OM}$. § 123, Ax. I

4. Also $\angle FGH=\angle ABC$, $\angle GHJ=\angle BCD$, etc. § 317

5. $\angle OFG=\angle OAB$, $\angle FGO=\angle ABO$. § 26

6. $\therefore$ since $\angle O$ is common, $\triangle FOG\sim\triangle AOB$. § 128

7. Similarly, $\triangle GOH\sim\triangle BOC$.

8. $\therefore \frac{FG}{AB}=\frac{OG}{OB}$ and $\frac{GH}{BC}=\frac{OG}{OB}$. Def. sim. $\triangle$s

9. $\therefore \frac{FG}{AB}=\frac{GH}{BC}$. Ax. I

10. Similarly, $\frac{GH}{BC}=\frac{HJ}{CD}$, etc.

11. $\therefore FGH\cdots\sim ABC\cdots$. Def. sim. poly.

391. Corollary 1. — *The area of a section of a pyramid parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude.*

The proof is left to the student.

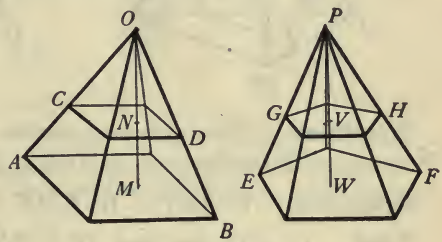
392. Corollary 2. — *If two pyramids have equal altitudes and equal bases, sections made by planes parallel to the bases at equal distances from the vertices are equal.*

For, $\frac{CD}{AB}=\frac{\overline{ON}^2}{\overline{OM}^2}$ and

$\frac{GH}{EF}=\frac{\overline{PV}^2}{\overline{PW}^2}$. Why?

Hence $\frac{CD}{AB}=\frac{GH}{EF}$. Why?

Hence $CD = GH$. Why?



EXERCISES

1. What part of the area of the base of a pyramid is the area of a section made by a plane which is parallel to the base and bisects the altitude?

2. The altitude of a pyramid is 16 in. and its base is a square 12 in. on a side. What is the area of a section parallel to the base whose distance from the vertex is 12 in.?

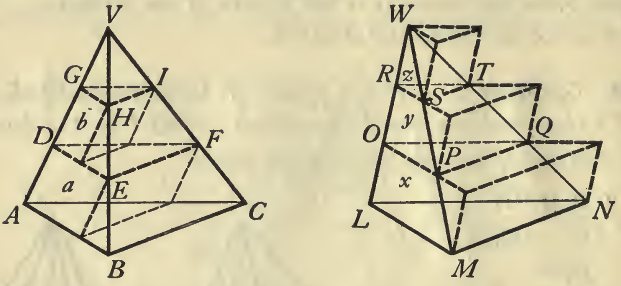
3. If two pyramids with equal altitudes are cut by planes parallel to the bases, and at equal distances from their vertices, the sections have the same ratio as the bases.

4. A pyramid 18 ft. high has a base containing 225 sq. ft. How far from the vertex must a plane be passed parallel to the base in order that the section may contain 64 sq. ft.?

5. At what point of the altitude of a pyramid should a plane be passed parallel to the base so that the section shall equal one half of the base?

6. The base of a pyramid is 10 yd. square. A plane parallel to the base and 9 yd. from the vertex cuts a section whose area is 36 sq. yd. Find the altitude of the pyramid.

393. Theorem. — *Two triangular pyramids having equal altitudes and equal bases are equal.*



Hypothesis. Triangular pyramids $V\text{–}ABC$ and $W\text{–}LMN$ have equal altitudes and equal bases.

Conclusion. $V\text{–}ABC = W\text{–}LMN$.

Proof. 1. Suppose that $W\text{–}LMN > V\text{–}ABC$.

2. Let each of the two equal altitudes be divided into n equal parts of length h. Through the points of division pass planes parallel to the bases, cutting the pyramids in $\triangle DEF$, $\triangle GHI$, etc., and $\triangle OPQ$, $\triangle RST$, etc., respectively.

3. On $\triangle DEF$, $\triangle GHI$, etc., as upper bases, and with AD, DG, etc., as lateral edges, construct prisms a, b, etc.

4. On $\triangle LMN$, $\triangle OPQ$, etc., as lower bases, and with LO, OR, etc., as lateral edges, construct prisms x, y, etc.

5. $\triangle DEF = \triangle OPQ$, $\triangle GHI = \triangle RST$, etc. § 392

6. $a = h \times \triangle DEF$, $y = h \times \triangle OPQ$, etc. § 369

7. $\therefore a = y$, $b = z$, etc. Ax. IV

8. But $V\text{–}ABC > a + b +$ etc., and $W\text{–}LMN < x + y + z +$ etc. Ax. X

9. $\therefore W\text{–}LMN - V\text{–}ABC < x + y + z +$ etc. $- (a + b +$ etc.$)$, or $W\text{–}LMN - V\text{–}ABC < x$, for the pyramids differ by less than the difference between the sums of the two sets of prisms.

10. Now, by increasing n indefinitely, and thus reducing

h indefinitely, x can be made less than any fixed or assigned quantity however small.

Hence, whatever value $W\text{-}LMN - V\text{-}ABC$ is assumed to have, x may be made less than $W\text{-}LMN - V\text{-}ABC$, which contradicts step 9.

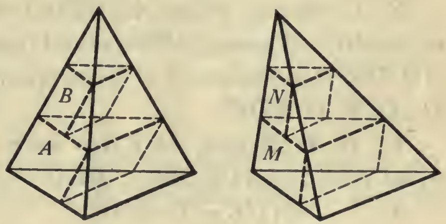
11. $\therefore$ the supposition is false, and $W\text{-}LMN$ is not greater than $V\text{-}ABC$.

12. Similarly, it may be shown that $V\text{-}ABC$ is not greater than $W\text{-}LMN$.

13. $\therefore$ $V\text{-}ABC = W\text{-}LMN$.

EXERCISES

1. Prove the theorem in § 393 by assuming that the volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed prisms of equal altitudes, if the number of prisms is indefinitely increased.



SUGGESTIONS. — Proceed as in § 393, except to construct inscribed prisms A, B, etc., and M, N, etc., with the sections as upper bases in both pyramids.

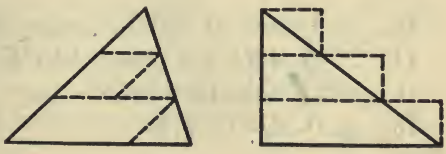
Show that $A + B +$ etc. $= M + N +$ etc.

Then apply § 275, (1).

2. A pyramid with a parallelogram as base is divided into two equal pyramids by a plane through the vertex and a diagonal of the base.

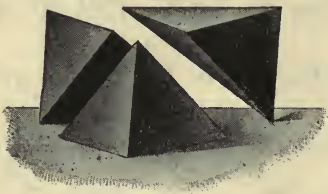
3. How may a given triangular pyramid be divided into four equal triangular pyramids?

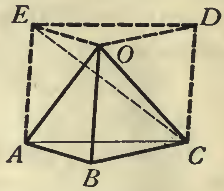
4. Prove the theorem of plane geometry that two triangles which have equal bases and equal altitudes are equal, by a proof similar to that in § 393.



SUGGESTION. — Inscribe a set of parallelograms in one triangle, and circumscribe a set of parallelograms about the other.

394. Theorem. — *The volume of a triangular pyramid is equal to one third of the product of its altitude and the area of its base.*





Hypothesis. $O\text{–}ABC$ is any triangular pyramid of which the altitude is h and the area of the base is a.

Conclusion. $O\text{–}ABC = \frac{1}{3}ha$.

Proof. 1. The altitude of $O\text{–}ABC$ is h and the area of the base is a. Hyp.

2. Construct prism $ABCEOD$ with base ABC and lateral edge OB. Through EO and OC pass a plane. Then prism $ABCEOD$ is composed of three triangular pyramids, $O\text{–}ABC$, $O\text{–}ACE$, $O\text{–}EDC$.

3. $O\text{–}ACE$ and $O\text{–}EDC$ have the same altitude and equal bases, $\triangle ACE$ and $\triangle EDC$. § 83

4. $\therefore O\text{–}ACE = O\text{–}EDC$. § 393

5. $O\text{–}EDC$ is identical with $C\text{–}EOD$.

6. $C\text{–}EOD$ and $O\text{–}ABC$ have the same altitude and equal bases, $\triangle EOD$ and $\triangle ABC$. § 349

7. $\therefore C\text{–}EOD = O\text{–}ABC$ § 393

8. $\therefore O\text{–}ACE = O\text{–}ABC$. Ax. I

9. Now $O\text{–}ABC + O\text{–}ACE + C\text{–}EOD =$ prism $ABCEOD$. Ax. X

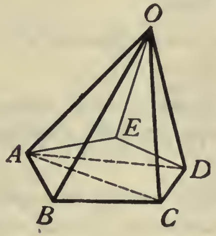
10. $\therefore$ 3 times $O\text{–}ABC =$ prism $ABCEOD$. Ax. XII

11. $\therefore O\text{–}ABC = \frac{1}{3}$ prism $ABCEOD$. Ax. V

12. But prism $ABCEOD = ha$. § 369

13. $\therefore O\text{–}ABC = \frac{1}{3}ha$. Ax. XII

395. Theorem. — *The volume of any pyramid is equal to one third of the product of its altitude and the area of its base.*



Hypothesis. $O\text{–}ABC \cdots$ is any pyramid of which the altitude is h and the area of the base is a.

Conclusion. $O\text{–}ABC \cdots = \frac{1}{3}\, ha$.

Suggestions. Upon what previous theorem may the proof be made to depend? How may $O\text{–}ABC \cdots$ be divided into triangular pyramids having the same altitude as $O\text{–}ABC \cdots$?

Write the proof in full.

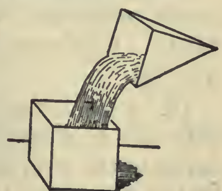
396. Corollary. — *The volume of any pyramid is equal to one third of the volume of a prism with an equal altitude and an equal base.*

The proof is left to the student.

EXERCISES

1. Saw out a triangular prism of wood. Then cut this prism into three triangular pyramids, and thus make a model to illustrate § 394.

2. Construct of cardboard a prism and a pyramid with equal bases and equal altitudes, leaving the base of the pyramid and one base of the prism open. Fill the pyramid level full of dry sand, and empty it into the prism. Repeat until the prism is full. Does this verify the corollary in § 396?



3. Two pyramids having equal altitudes and equal bases are equal.

4. The volumes of any two pyramids with equal altitudes are to each other as the areas of their bases.

5. The volumes of any two pyramids with equal bases are to each other as their altitudes.

6. The volume of a regular pyramid is equal to the lateral area multiplied by one third of the perpendicular distance from the center of the base to any lateral face.

7. Where must a plane be passed through a given pyramid parallel to the base so that the pyramid cut off shall equal one ninth of the given pyramid?

8. A triangular pyramid can be constructed equal to any given pyramid whose base is any polygon.

9. The volume of a pyramid equals the product of the altitude and the area of a section parallel to the base how far from the vertex?

10. The line-segments joining the center of a cube to the four vertices of one face are the lateral edges of a regular quadrangular pyramid whose volume is one sixth that of the cube.

11. The altitude of a pyramid is 6 ft., and its base is a rectangle 5 ft. long and 4 ft. wide. Find the volume.

12. The altitude of a pyramid is 18 in., and its base is a trapezoid whose parallel sides are 6 in. and 14 in. respectively, and the distance between them 12 in. Find the volume.

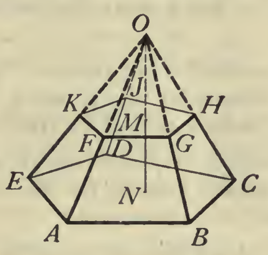
13. Find the volume of a regular pyramid whose altitude is 24 ft. and base a triangle each side of which is 12 ft.

14. Find the volume of a regular hexagonal pyramid whose altitude is 40 in. and each side of whose base is 15 in.

15. A farmer has a rick of corn piled out of doors that is 12 ft. wide at the bottom and 35 ft. long. It tapers to a point 10 ft. high in the middle. How many bushels does it contain? (Count 2 bu. for each 5 cu. ft.)

16. A farmer has a corn crib 12 ft. wide and 16 ft. long. It is filled with corn to a depth of 6 ft. around the walls, and heaped up in the center in the form of a pyramid to a total depth of 9 ft. How many bushels does it contain?

397. Theorem. — *The volume of a frustum of a pyramid is equal to one third of the product of its altitude and the sum of the areas of the bases and the mean proportional between them.*



Hypothesis. B = area of larger base, b = area of smaller base, and h = altitude of frustum AH of any pyramid.

Conclusion. The volume of $AH = \frac{1}{3}h(B + b + \sqrt{Bb})$.

Proof. 1. B = area of larger base, b = area of smaller base, h = altitude of frustum AH of any pyramid. Hyp.

2. Let $O\text{-}ABC \cdots$ be the pyramid of which AH is a frustum, and let its altitude ON meet base FH at M.

3. Volume of $O\text{-}ABC \cdots = \frac{1}{3}B(h + OM)$, volume of $O\text{-}FGH \cdots = \frac{1}{3}b(OM)$. § 395

4. $\therefore$ volume of $AH = \frac{1}{3}B(h + OM) - \frac{1}{3}b(OM)$ or $\frac{1}{3}hB + \frac{1}{3}OM(B - b)$. Ax. III

5. But $\dfrac{B}{b} = \dfrac{\overline{ON}^2}{\overline{OM}^2}$, and hence $\dfrac{\sqrt{B}}{\sqrt{b}} = \dfrac{ON}{OM}$. § 391, Ax. VI

6. $\therefore \dfrac{\sqrt{B} - \sqrt{b}}{\sqrt{b}} = \dfrac{ON - OM}{OM}$ or $\dfrac{h}{OM}$. § 118, (6)

7. $\therefore OM(\sqrt{B} - \sqrt{b}) = h\sqrt{b}$. Clearing fract.

8. $\therefore$ multiplying both members by $\sqrt{B} + \sqrt{b}$,

$$OM(B - b) = h(\sqrt{Bb} + b). \qquad \text{Ax. IV}$$

9. $\therefore$ volume of $AH = \frac{1}{3}hB + \frac{1}{3}h(\sqrt{Bb} + b)$. Ax. XII

10. $\therefore$ volume of $AH = \frac{1}{3}h(B + b + \sqrt{Bb})$. Factoring

EXERCISES

1. The frustum of a pyramid is equal to the sum of three pyramids having a common altitude equal to that of the frustum, and for bases the bases of the frustum and a mean proportional between them, respectively.

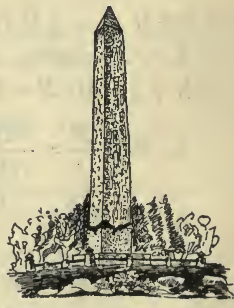
2. Show that the formula for the volume of a pyramid in § 395 may be obtained from the formula for the volume of a frustum of a pyramid in § 397 by making the smaller base of the frustum zero.

3. Show that the formula for the volume of a frustum of a pyramid in § 397 would reduce to the formula for the volume of a prism if the bases were made equal.

4. Find the volume of a frustum of a regular quadrangular pyramid whose altitude is 12 in. and the sides of whose bases are 16 in. and 8 in., respectively.

5. Find the volume of a frustum of a triangular pyramid whose altitude is 16 ft. and the bases equilateral triangles whose sides are 10 ft. and 4 ft., respectively.

6. The base of a granite monument is a frustum of a pyramid, of which the bases are squares whose sides are 6 ft. and 5 ft., respectively, and of which the altitude is $3\frac{1}{2}$ ft. Granite weighs 170 lb. to the cubic foot. Find the total weight of the base.



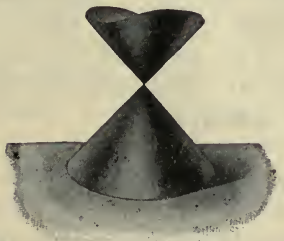
7. One of "Cleopatra's Needles," quarried in one piece, floated down the Nile, and erected at Heliopolis, Egypt, about 1500 B.C., has been removed to Central Park, New York. It is a frustum of a quadrangular pyramid, 64 ft. high, 8 ft. square at the base, and 5 ft. square at the top, surmounted by a pyramid 7 ft. high. It is one of the largest stones ever quarried in a single piece. Counting 170 lb. to the cubic foot, compute its weight.

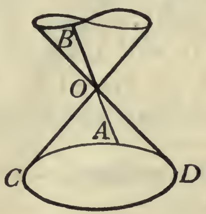
8. A factory chimney is to be constructed in the form of a frustum of a pyramid with a square base. The height is to be 140 ft., and the sides of the bases are to be 24 ft. and 12 ft., respectively. The flue is to be 6 ft. square throughout the entire height. How many thousands of bricks, 2 in. by 4 in. by 8 in., must be ordered to build it, allowing 10 % of the space for mortar?

9. A grain elevator is in the form of a frustum of a pyramid 32 ft. high. The bases are square, with sides 13 ft. and 6 ft., respectively. Allowing $\frac{4}{5}$ bu. to 1 cu. ft., find how many bushels of corn it will hold.

10. A fruit grower sells cherries in boxes 5 in. square at the top, 4 in. square at the bottom, and 3 in. deep. Allowing 67.2 cu. in. to a quart, does one of these boxes hold a quart of cherries?

398. Cones. — A **conical surface** is a surface traced by a moving straight line which constantly intersects a fixed plane curve and passes through a fixed point not in the plane of the curve. The fixed point is called the **vertex**. The moving line in any position is called an **element** of the surface.





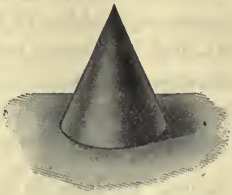
Thus, if line AB moves so that it constantly intersects the fixed plane curve ACD and constantly passes through the fixed point O, not in the plane of ACD, it traces a conical surface. O is the vertex, and AB, in any one position, is an element.

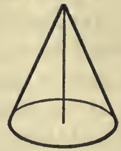
A conical surface consists of two parts, which are traced by the two parts into which the moving line is divided at the vertex. These two parts are called the **nappes** of the surface.

A **cone** is a solid bounded by a portion of one nappe of a conical surface and that part of a plane cutting all elements of the conical surface which lies within the surface. The portion of the plane is called the **base**, and the conical surface is called the **lateral surface** of the cone.

The perpendicular distance from the vertex to the base of a cone is called its **altitude**.

A **circular cone** is one whose base is inclosed by a circle.





RIGHT CIRCULAR CONE

The line-segment joining the vertex of a circular cone to the center of the base is called the **axis** of the cone.

A **right circular cone** is a circular cone whose axis is perpendicular to the base.

A right circular cone is called also a **cone of revolution**, for if a right triangle is revolved about one of its legs as an axis, it will trace such a cone. Two cones of revolution which are traced by revolving similar right triangles about homologous legs as axes are called **similar cones**.

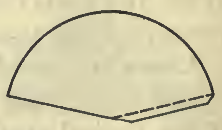
399. Corollary. — *The elements of a right circular cone are equal.*

The proof is left to the student. Write the proof in full.

400. Slant height of a cone. — The length of an element of a right circular cone is called the **slant height** of the cone.

EXERCISES

1. Draw on paper or pliable cardboard, a sector of a circle, as shown in the drawing. Make the radius 4 in. and the central angle less than 180°. Cut out the pattern, and by folding and pasting, make a model of the lateral surface of a right circular cone.

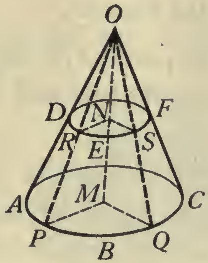


2. The altitude of a right circular cone is 12 ft. and the radius of the base is 5 ft. Find the slant height.

3. The slant height of a right circular cone is 40 in. and the radius of the base is 18 in. Find the altitude.

4. The slant heights of two similar cones of revolution have the same ratio as their altitudes or as the radii of their bases.

401. Theorem. — *A section of the lateral surface of any circular cone made by a plane parallel to the base is a circle.*



Hypothesis. $O\text{-}ABC$ is a circular cone; DEF is a section of the lateral surface made by a plane parallel to the base.

Conclusion. DEF is a circle.

Proof. 1. Draw axis OM, intersecting plane DEF at N.

2. Let R and S be any two points on DEF. Draw elements OP and OQ through R and S, respectively. Let planes OMP and OMQ intersect plane of base in MP and MQ and plane of DEF in NR and NS, respectively.

3. Then $NR \parallel MP$. § 315

4. $\therefore \angle NRO = \angle MPO$ and $\angle RNO = \angle PMO$. § 26

5. $\angle RON = \angle POM$ identically.

6. $\therefore \triangle ORN \sim \triangle OPM$. § 128

7. $\therefore \dfrac{RN}{PM} = \dfrac{ON}{OM}$. Def. sim. poly.

8. Similarly, $\dfrac{SN}{QM} = \dfrac{ON}{OM}$.

9. $\therefore \dfrac{RN}{PM} = \dfrac{SN}{QM}$. Ax. I

10. But $PM = QM$. § 151, (2)

11. $\therefore RN = SN$. Clearing fract.

12. $\therefore$ all points on DEF are equidistant from N, and hence DEF is a circle. Def. ⊙

402. Corollary 1. — *The axis of a circular cone passes through the center of every section parallel to the base.*

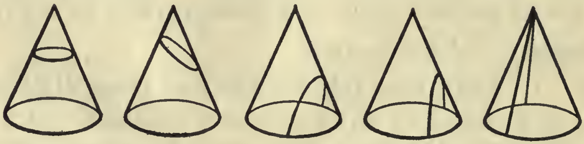
The proof is left to the student.

403. Corollary 2. — *The parts of the elements of a right circular cone included between the base and a plane parallel to the base are equal.*

The proof is left to the student.

404. Conic sections. — A section of the lateral surface of a right circular cone by a plane is called a **conic section**.

There are five possible forms of conic sections, as follows:



(1) (2) (3) (4) (5)

If the cutting plane is parallel to the base, the section was proved in § 401 to be a *circle*, as shown in figure (1).

If the cutting plane intersects all elements but is not parallel to the base, the section is an oval called an *ellipse*, as shown in figure (2). The ellipse is of great interest and importance. The orbits of the earth and all other planets are ellipses.

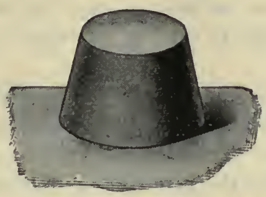
If the cutting plane is parallel to an element, the section is a *parábola*, as shown in figure (3). The path of a projectile, such as a stone thrown upward, or a bullet from a gun, is a parabola.

If the cutting plane is parallel to the axis, the section is an *hyperbola*, as shown in figure (4).

If the cutting plane passes through the vertex, the section is *two straight lines*, as shown in figure (5).

405. Frustum of a cone. — A **frustum of a cone** is the portion of the cone lying between the base and a plane parallel to the base.

The base of the cone and the section made by the plane are called the **bases** of the frustum.



FRUSTUM OF CONE

The perpendicular distance between the bases is called the **altitude** of the frustum.

The length of an element included between the bases is called the **slant height** of the frustum.

EXERCISES

1. Every section of a cone made by a plane passing through the vertex is a triangle.

SUGGESTION. — Prove that the intersections of the plane and the conical surface are elements.

2. Every section of a frustum of a cone made by a plane passing through an element is a trapezoid.

3. Find the locus of the centers of all circular sections made by planes parallel to the base of a circular cone.

4. The radii of the bases of a frustum of any circular cone have the same ratio as the distances of the bases from the vertex of the cone.

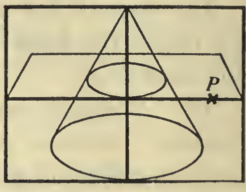
5. Two sections of a circular cone made by planes parallel to the base are to each other as the squares of their distances from the vertex.

6. On paper or pliable cardboard, draw a figure similar to the adjoining figure. By cutting out the pattern, folding and pasting, make a model of the conical surface of a frustum of a right circular cone.

7. The altitude of a frustum of a right circular cone is 8 in., and the radii of the bases are 4 in. and 6 in., respectively. Find the slant height.

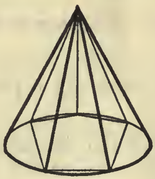
8. The altitude, slant height, and radius of the larger base of a frustum of a right circular cone are 6 in., 8 in., and 10 in., respectively. Find the radius of the smaller base.

9. A plane determined by an element of a circular cone and a tangent to the base contains no point of the surface without that element. Such a plane is said to be *tangent* to the cone.

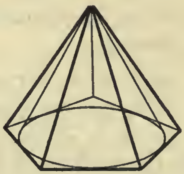


SUGGESTION. — Let P be any point of the plane without the element. Draw a plane through P parallel to the base. Prove that this plane intersects the given plane in a tangent to the section of the cone.

406. Inscribed and circumscribed pyramids and cones. — A pyramid is **inscribed in a circular cone** when the base of the pyramid is inscribed in the base of the cone and the two have the same vertex.



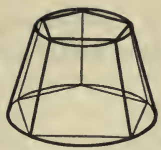
INSCRIBED PYRAMID



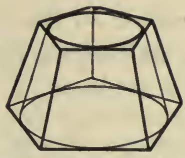
CIRCUMSCRIBED PYRAMID

A pyramid is **circumscribed about a circular cone** when the base of the pyramid is circumscribed about the base of the cone and the two have the same vertex. It is evident that when a regular pyramid is circumscribed about a right circular cone the slant height of the pyramid is equal to the slant height of the cone.

A frustum of a pyramid is **inscribed in a frustum of a circular cone** when its bases are inscribed in the bases of the frustum of a cone.



Inscribed Frustum of Pyramid



Circumscribed Frustum of Pyramid

A frustum of a pyramid is **circumscribed about a frustum of a circular cone** when its bases are circumscribed about the bases of a frustum of a cone. It is evident that when a frustum of a regular pyramid is circumscribed about a frustum of a right circular cone the slant height of the frustum of the pyramid is equal to the slant height of the frustum of the cone.

407. Limits. — The following principles of limits are assumed:

(1) *The lateral area of a right circular cone (frustum of a right circular cone) is the limit approached by the lateral area of an inscribed or circumscribed regular pyramid (frustum of a regular pyramid) as the number of faces is indefinitely increased.*

(2) *The volume of a circular cone (frustum of a circular cone) is the limit approached by the volume of an inscribed or circumscribed pyramid (frustum of a pyramid) whose base is a regular polygon as the number of faces is indefinitely increased.*

408. Principles of limits. — The following general principles of limits are assumed, as were similar principles in § 275 :

(1) *The limit of the sum of two or more variables is the sum of their limits.*

(2) *The limit of the product of two or more variables is the product of their limits.*

(3) *The limit of any principal root of a variable is that root of its limit.*

409. Theorem. — *The lateral area of a right circular cone is equal to one half of the product of its slant height and the circumference of its base.*

Hypothesis. $S =$ the lateral area, $C =$ the circumference of the base, and $L =$ the slant height of a right circular cone.

Conclusion. $S = \frac{1}{2} LC$.

Proof. 1. $S =$ the lateral area, $C =$ the circumference of the base, and $L =$ the slant height of a right circular cone. Hyp.

2. Let $s =$ the lateral area of a circumscribed regular pyramid, and let $p =$ the perimeter of its base. Then $L =$ the slant height of the pyramid also. § 406

3. $\therefore s = \frac{1}{2} Lp$. § 388

4. Now, if the number of lateral faces of the pyramid is increased indefinitely, $s \doteq S$ and $p \doteq C$. § 407, § 274

5. $\therefore \frac{1}{2} Lp \doteq \frac{1}{2} LC$. § 275, (2)

6. $\therefore S = \frac{1}{2} LC$. § 275, (1)

Using different letters, write out the proof without referring to the book.

410. Corollary. — *If S is the lateral area of a right circular cone, L the slant height, and R the radius of the base, then*

$$S = \pi RL.$$

The proof is left to the student.

EXERCISES

1. Can the theorem in § 409 be proved by inscribing a regular pyramid in the cone?

2. Compare the lateral areas of a right circular cylinder and a right circular cone with equal bases and equal altitudes when the slant height of the cone is equal to the diameter of the base.

3. Write the formula for the total area of a right circular cone.

4. Since the lateral surface of a cone of revolution may be produced by rolling up a sector of a circle, prove the theorem in § 409 by use of § 284.

5. The slant height of a right circular cone is 8 ft. and the radius of the base 6 ft. Find the lateral area. The total area.

6. The altitude of a right circular cone is 12 in., and the diameter of the base 14 in. Find the slant height. The lateral area. The total area.

7. The altitude of a right circular cone is 18 ft., and the slant height 24 ft. Find the radius of the base. The lateral area. The total area.

8. How many square yards of canvas will be required to make a conical tent whose altitude is 12 ft., and diameter of base 12 ft.?

9. A conical slate roof of a tower is 20 ft. high, and the diameter of its base is 18 ft. Find the area of the slate.

10. Find the area of the surface generated by revolving a right triangle about its hypotenuse as axis, the legs of the triangle being 3 in. and 4 in., respectively.

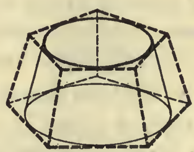
11. If a cone of revolution is formed by revolving an equilateral triangle about one of the altitudes as an axis, the lateral area equals twice the area of the base.

SUGGESTION. — Express the lateral area of the cone in terms of a side of the triangle.

12. An equilateral triangle each of whose sides is 6 in. is revolved about a straight line through a vertex and parallel to the opposite side as axis. Find the area of the whole surface traced by the three sides of the triangle.

13. A silo which is 18 ft. in diameter has a conical roof. The rafters are at an angle of 30° with horizontal, and project 1 ft. over the eaves. Find the area of the roof.

411. Theorem. — *The lateral area of a frustum of a right circular cone is equal to one half of the product of its slant height and the sum of the circumferences of its bases.*



Hypothesis. $S =$ the lateral area, $C =$ circumference of larger base, $c =$ circumference of smaller base, and $L =$ slant height of a frustum of a right circular cone.

Conclusion. $S = \frac{1}{2} L(C + c)$.

Proof. 1. $S =$ lateral area, C and c respectively $=$ circumferences of bases, and $L =$ slant height of frustum of right circular cone. Hyp.

2. Let $s =$ lateral area of a circumscribed frustum of a regular pyramid, and let P and p respectively $=$ perimeters of its bases. Then $L =$ its slant height also. § 406

3. $\therefore s = \frac{1}{2} L(P + p)$. § 389

4. Now, if the number of lateral faces is increased indefinitely, $s \doteq S$, $P \doteq C$, and $p \doteq c$. § 407

5. $\therefore (P + p) \doteq (C + c)$. § 408, (1)

6. $\therefore \frac{1}{2} L(P + p) \doteq \frac{1}{2} L(C + c)$. § 275, (2)

7. $\therefore S = \frac{1}{2} L(C + c)$. § 275, (1)

412. Corollary. *If S is the lateral area of a frustum of a right circular cone, L the slant height, and R and r respectively the radii of the bases, then*

$$S = \pi L(R + r).$$

The proof is left to the student.

EXERCISES

1. The lateral area of a frustum of a right circular cone is equal to the perimeter of a mid-section made by a plane parallel to the bases, multiplied by the slant height.

2. Show that the formula for computing the lateral area of a right circular cone is obtained from the formula for the lateral area of a frustum of a right circular cone by making one of the bases of the frustum reduce to a point.

3. Write the formula for the total area of a frustum of a right circular cone.

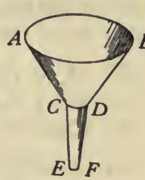
4. If a trapezoid, one of whose non-parallel sides is perpendicular to the bases, is revolved about this side as axis, the lateral area of a frustum of a cone traced equals 2π times the product of the other non-parallel side and the line-segment joining the middle points of the non-parallel sides.

SUGGESTION. — See Exercise 1 above.

5. The slant height of a frustum of a right circular cone is 14 in. and the diameters of the bases are 16 in. and 10 in., respectively. Find the lateral area. The total area.

6. The altitude of a frustum of a right circular cone is 8 ft., and the radii of the bases 4 ft. and 9 ft., respectively. Find the slant height. The lateral area. The total area.

7. A funnel is made of tin, in which the diameter AB is 8 in., the diameter CD is 1 in., the diameter EF is $\frac{1}{2}$ in., the length AC is 5 in., and the length CE is 4 in. Find the amount of tin required to make it, not allowing for the seams.



8. Find the amount of sheet metal required to make a lot of 1000 pails, each 10 in. deep, 8 in. in diameter at the bottom, and 11 in. in diameter at the top, not allowing for seams nor waste in cutting.

9. Given the diameters and the altitude of a frustum of a right circular cone, explain how to draw the pattern by which a piece of sheet metal must be cut out in order to be rolled up and made into the required frustum.

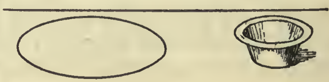
SUGGESTIONS. — Find the slant height. Let $x =$ the radius of the inner arc of the pattern. Then $x +$ the slant height $=$ the radius of the

outer arc. These radii are proportional to the circumferences of the bases of the frustrum. Solve the proportion for x.

10. A tinner wishes to make a coffee pot which shall be 7 in. deep, 4 in. in diameter at the top, and 6 in. in diameter at the bottom. Allowing a quarter of an inch for a seam, cut from paper a pattern for the conical surface.

NOTE. — For explanation of the terms used, and of the process involved in the manufacture of articles as indicated in the following problems, see Problem 9, page 352.

11. A stew pan is to be made in the form of a frustum of a right circular cone, $6\frac{1}{2}$ in. wide at the bottom, $8\frac{1}{2}$ in. wide at the top, 3 in. deep, and to have a flange $\frac{1}{2}$ in. wide at the top. Find the diameter of the blank from which it must be made.

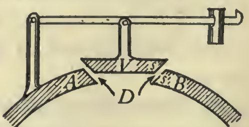


SUGGESTION. — The area of the pan consists of three parts: the bottom; the side, which is the lateral area of a frustum of a cone; and the flange, which is the difference between the areas of two circles. The sum of these must equal the area of the blank. Indicate the factors, and avoid as much multiplication as possible.

12. A tin dish pan is to be made 13 in. wide at the bottom, 17 in. wide at the top, 6 in. deep, and have a $\frac{1}{2}$-in. flange at the top. Find the diameter of the blank from which it must be made.

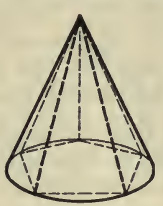
13. A tin pie pan is $6\frac{3}{4}$ in. in diameter at the bottom, 8 in. in diameter at the top, 1 in. deep, and has a $\frac{1}{2}$-in. flange at the top. Find the diameter of the blank from which it is made.

14. A safety valve of an engine has a valve V in the form of a frustum of a cone, which closes the opening D in the boiler AB. The diameter of the opening D is 2 in. The conical surface of the valve is inclined at 45°. The "effective area," or area of the opening through which the steam escapes when the valve V is lifted, is evidently the lateral area of a frustum of a cone whose slant height is ss.



If the valve V is lifted through a height of $\frac{1}{4}$ in., find the effective area.

413. Theorem. — *The volume of a circular cone is equal to one third of the product of its altitude and the area of its base.*



Hypothesis. $V =$ volume, $H =$ altitude, and $B =$ the area of the base of a circular cone.

Conclusion. $V = \frac{1}{3} HB$.

Suggestions. Inscribe a pyramid whose base is a regular polygon.

What formula expresses the volume of the pyramid?

The formula $V = \frac{1}{3} HB$ may be proved from this by § 275, (1), if $\frac{1}{3} HB$ is first proved to be the limit of what?

Write the proof in full.

414. Corollary. — *If V is the volume of a circular cone, H the altitude, and R the radius of the base, then*

$$V = \frac{1}{3} \pi R^2 H.$$

The proof is left to the student.

EXERCISES

1. The volume of a circular cone is one third of the volume of a circular cylinder having the same base and altitude.

2. Prove the theorem in § 413 by circumscribing about the cone a pyramid with a regular base.

3. Find the volume of a circular cone of which the altitude is 16 in. and the diameter of the base 12 in.

4. Find the volume of a right circular cone the radius of whose base is 6 in. and whose slant height is 20 in.

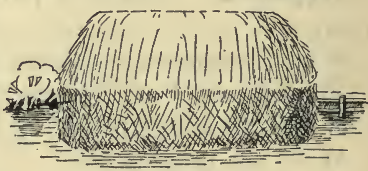
5. A fruit raiser has a round pile of apples that is 8 ft. across at the bottom, and tapers to a point 5 ft. high at the middle. Allowing 3 bu. to 4 cu. ft., how many bushels are there in the pile?

6. A farmer has a pile of ear corn approximately in the form of a cone whose height is 10 ft. and width at the bottom 20 ft. Allowing 2 bu. to 5 cu. ft., how many bushels does it contain?

7. Crushed stone falling from a stone crusher forms a round pile 30 ft. around at the bottom and 5 ft. high at the center. How many cubic yards of crushed stone are there in the pile?

8. A farmer wishes to know how many tons of hay there are in a stack in the form of a cylinder 18 ft. in diameter and 8 ft. high, surmounted by a cone 10 ft. high. Allowing 512 cu. ft. to the ton, find the amount of hay in the stack.

9. A farmer has a rick of hay 36 ft. long and 16 ft. wide. The lower part is made up of a parallelopiped with half cylinders at its ends, and is 8 ft. high. The part surmounting this tapers to a line 20 ft. long and 18 ft. above the ground, and is made up of a triangular prism with half cones at its ends. Find the number of tons in it, allowing 512 cu. ft. to the ton.



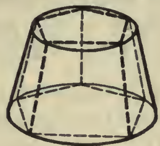
10. A hill, approximately in the form of a cone, is to be removed and the ground leveled down even with its base, and laid off into building lots. The diameter of the base is 280 ft., and the highest point is 12 ft. above the level. How many loads (cubic yards) of earth must be removed?

11. In the center of a building lot is a piece of elevated ground approximately in the form of a cone. The elevated part is to be cut down and used to raise the level of the surrounding ground. The lot is 100 ft. wide and 120 ft. long. The height of the cone is 5 ft. and the diameter of its base 60 ft. Find how much the elevated part must be cut, and how much the rest of the lot must be filled, to bring it to a level.

12. Given the lateral area S of a right circular cone, and the radius R of the base. Find the volume V.

13. Given the total area T of a right circular cone, and the lateral area S. Find the volume V.

415. Theorem. — *The volume of a frustum of a circular cone is equal to one third of the product of its altitude and the sum of the areas of the bases and the mean proportional between them.*



Hypothesis. $V =$ volume, $H =$ altitude, and B and b respectively $=$ the areas of the bases of a frustum of a circular cone.

Conclusion. $V = \frac{1}{3} H(B + b + \sqrt{Bb})$.

Suggestions. Inscribe a frustum of a pyramid whose base is a regular polygon.

What formula expresses the volume of the frustum of a pyramid?

The formula $V = \frac{1}{3} H(B + b + \sqrt{Bb})$ may be proved from this by § 275, (1), if $\frac{1}{3} H(B + b + \sqrt{Bb})$ is first proved to be the limit of what?

Write the proof in full.

416. Corollary. — *If V is the volume of a frustum of a circular cone, H its altitude, and R and r respectively the radii of the bases, then*

$$V = \frac{1}{3} \pi H(R^2 + r^2 + Rr).$$

The proof is left to the student.

EXERCISES

1. The volume of a frustum of a circular cone is equal to the sum of the volumes of the three circular cones having a common altitude equal to the altitude of the frustum, and for bases the two bases of the frustum and a mean proportional between them, respectively.

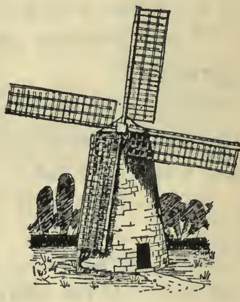
2. Find the volume of a frustum of a circular cone of which the altitude is 4 in. and the diameters of the bases 6 in. and 5 in., respectively.

3. Find the volume of a frustum of a right circular cone of which the slant height is 32 in. and the radii of the bases 16 in. and 28 in. respectively.

4. Does a liquid measure with a diameter of 8 in. at the top, a diameter of $13\frac{1}{2}$ in. at the bottom, and $12\frac{5}{8}$ in. deep, contain 5 gal.? (1 gal. = 231 cu. in.)

5. A Sanford's Ink bottle is in the form of a frustum of a cone whose height is $1\frac{3}{4}$ in., and the diameters at the top and bottom $1\frac{1}{4}$ in. and 2 in., respectively. How many of these bottles can be filled with a gallon of ink?

6. The base of a marble column is a frustum of a cone. The height is 1 ft. 6 in., and the diameters of the bases 6 ft. and 5 ft. 6 in., respectively. Allowing 170 lb. to the cubic foot, find its weight.



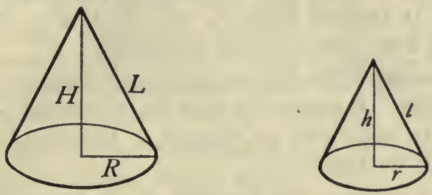
7. A Dutch windmill, in the shape of a frustum of a cone, is 30 ft. high. The outer diameters at the bottom and top are 15 ft. and 12 ft., respectively; and the inner diameters at the bottom and top are 12 ft. and 9 ft., respectively. How many cubic yards of stone were required to build it?

8. The chimney of a factory is to be 100 ft. high; the outer diameters at the bottom and top are to be 18 ft. and 10 ft., respectively; and the flue is to be 6 ft. in diameter throughout. How many thousands of bricks, 2 in. by 4 in. by 8 in., must the contractor order for it, allowing 10% for mortar?

9. A churn, in the form of a frustum of a cone, is 28 in. high, 10 in. in diameter at the bottom, and 8 in. in diameter at the top. How many gallons of cream will it hold?

10. A cake pan is 10 in. in diameter at the top and 8 in. in diameter at the bottom, and its depth is 3 in. Rising from the center of the bottom is a stem in the form of a frustum of a cone $1\frac{1}{2}$ in. in diameter at the bottom and 1 in. in diameter at the top, which makes the hole in the cake. How many cubic inches does the pan hold?

417. Theorem. — *The lateral or the total areas of two similar cones of revolution are to each other as the squares of the slant heights, the altitudes, or the radii of the bases; and their volumes are to each other as the cubes of the slant heights, the altitudes, or the radii of the bases.*



Hypothesis. S and s are the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases, of two similar cones of revolution.

Conclusion. $\frac{S}{s}=\frac{T}{t}=\frac{L^2}{l^2}=\frac{H^2}{h^2}=\frac{R^2}{r^2}$; $\frac{V}{v}=\frac{L^3}{l^3}=\frac{H^3}{h^3}=\frac{R^3}{r^3}$.

Proof. The proof is left to the student. Proceed as in § 379 and § 382. Write the proof in full.

EXERCISES

1. The lateral area of a cone of revolution is 144 sq. in. What is the lateral area of a similar cone whose altitude is $\frac{3}{4}$ that of the given cone?

2. The altitude of a circular cone is 8 ft. How far from the vertex must a plane be passed parallel to the base to cut off $\frac{1}{16}$ of the cone from the vertex?

3. To divide a circular cone into halves by a plane parallel to the base, how far from the vertex must the plane be passed?

4. From a cone of revolution of which the slant height is 24 ft. and of which the diameter of the base is 12 ft., a cone of which the slant height is 8 ft. is cut off by a plane parallel to the base. Find the lateral area and the volume of the cone cut off; of the frustum.

MISCELLANEOUS EXERCISES

1. A lamp shade, made of a wire frame covered with silk, is in the form of a frustum of a right circular cone whose slant height is 9 in. and the radii of whose bases are 6 in. and $4\frac{1}{2}$ in., respectively. If one half extra is allowed for fullness in shirring the silk, how much silk is required for the lamp? If the silk is 27 in. wide, find to an eighth of a yard how much of a yard is required.

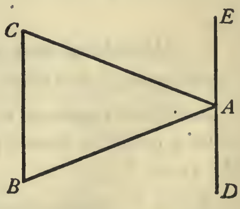
2. The lateral area of a right circular cone is twice the area of the base. Compare the slant height and the radius. At what angle does an element of the cone meet the plane of the base?

3. If r is the radius of the base of a right circular cone and h the slant height, prove that the ratio of the area of the base to the lateral area is $\frac{r}{h}$.

4. If a right circular cone and a right circular cylinder have equal bases and the slant height of the cone is equal to the altitude of the cylinder, prove that the entire area of the cylinder is equal to twice the entire area of the cone.

5. An ice cream dipper is in the form of a right circular cone whose diameter is $2\frac{3}{4}$ in. and altitude $2\frac{3}{4}$ in. It is filled level full at each serving. How many servings can be obtained from a gallon (231 cu. in.) of ice cream? What is received for a gallon at 5¢ a serving?

6. Triangle ABC is an isosceles triangle with equal sides AB and AC. Line DE is drawn through A parallel to BC, and $\triangle ABC$ is revolved about DE as an axis. Find the volume of the figure traced by $\triangle ABC$.

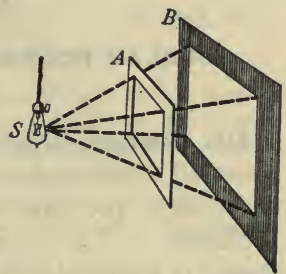


7. A conical glass beaker, graduated for measuring liquids, holds a quart. At what point of the scale must the mark be placed to measure a pint? To measure a gill?

8. A farmer has two similar piles of corn, each in the form of a right circular cone. The diameters of the two piles are 4 ft. and 18 ft., respectively. He measures the small pile, and finds that it contains

10 bu. How can he estimate, without measuring, the number of bushels in the large pile? How many bushels are there in the large pile?

9. Two boards, A and B, are placed in parallel vertical positions on a table. A has a square opening cut in it. A lamp is placed at S. Light from this lamp passes through the opening in A, and illuminates part of B. Show, by aid of this diagram, that the intensity of the light falling upon a given surface is inversely proportional to the square of the distance of the surface from the source of light.



10. The altitude of each of two pyramids whose bases are in the same plane is 16 ft. The base of one is a square whose side is 12 ft., and the base of the other is a regular hexagon whose side is 8 ft. The area of the section of the first made by a plane parallel to the base is 81 sq. ft. What is the area of the other pyramid made by the same plane?

11. A fruit vender has a load of apples in a wagon bed that is 3 ft. wide, 1 ft. deep, and 10 ft. long. The apples fill the bed and are piled up in approximately a pyramid to a point 1 ft. higher than the top of the bed. How many bushels are there? (Count 3 bu. to 4 cu. ft.)

12. A triangular piece of ground ABC is to be leveled. The side AB is level, and the corner C is 12 ft. higher than AB. Find how much the ground must be cut down at C and how much it must be filled along AB to make it level.

13. Find the volume of a regular quadrangular pyramid whose altitude is 20 ft. and slant height 28 ft.

14. Find the volume of a triangular pyramid each edge of which is 4 ft.

15. The slant height of a frustum of a regular quadrangular pyramid is 18 in., and the sides of the bases are 20 in. and 12 in., respectively. Find its volume.

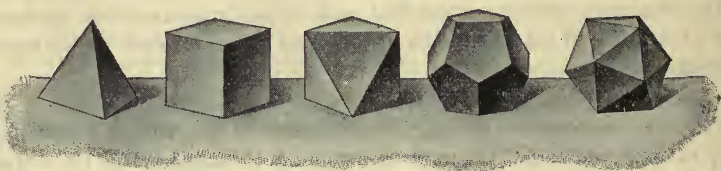
16. The lateral edge of a frustum of a regular hexagonal pyramid is 6 ft., and the sides of the bases are 8 ft. and 4 ft., respectively. Find its volume.

CHAPTER XV

REGULAR POLYEDRONS. SIMILAR POLYEDRONS. PRISMATOIDS

418. Regular polyedrons. — A **regular polyedron** is a polyedron all of whose faces are congruent regular polygons and all of whose polyedral angles are equal.

Models of five regular polyedrons are as follows. That these are the only kinds possible is proved in § 419.



TETRAEDRON HEXAEDRON OCTAEDRON DODECAEDRON ICOSAEDRON

419. Theorem. — *There cannot be more than five kinds of regular polyedrons.*

Proof. 1. Each angle of an equilateral triangle is 60°; of a square, 90°; of a regular pentagon, 108°; etc. § 101

2. ∴ not more than *three* kinds of regular polyedrons are possible with equilateral triangles as faces (*tetraedron, octaedron, icosaedron*); for $3 \times 60°$, $4 \times 60°$, or $5 \times 60°$ is less than 360°, while $6 \times 60°$ or any greater multiple of 60° is not less than 360°. § 343

3. Similarly, not more than *one* kind of regular polyedron is possible with squares as faces (*hexaedron* or *cube*); for $3 \times 90°$ is less than 360°, while $4 \times 90°$ or any greater multiple of 90° is not less than 360°.

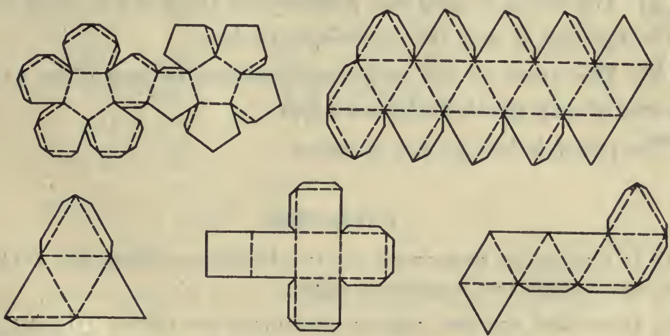
4. Similarly, not more than *one* kind of regular polyedron is possible with regular pentagons as faces (*dodecaedron*).

5. No regular polyedron is possible with regular hexagons or regular polygons of more sides as faces, because the sum of the face angles at each vertex would not be less than 360°.

6. ∴ only five regular polyedrons are possible.

NOTE. — The regular polyedrons were studied by the ancient Greeks. The members of the Pythagorean school knew that there were five of these solids. It is said that Hippasus (about 470 B.C.), a Pythagorean who discovered the regular dodecaedron, was drowned for announcing his discovery, because the Pythagoreans were a secret society, and the members were pledged to refer the honor of any new discovery to Pythagoras, the founder.

420. Construction of models of regular polyedrons. — Models of the five regular polyedrons may be constructed of cardboard by first drawing patterns as shown below, then cutting out the patterns, folding along the dotted lines, and pasting.



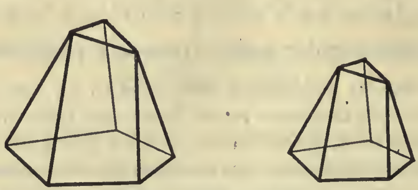
EXERCISES

1. How many faces has a regular tetraedron? Hexaedron? Octaedron? Dodecaedron? Icosaedron?

2. Construct cardboard models of one or more of the five regular polyedrons as suggested in § 420.

3. Prove that a regular hexaedron has a center, *i.e.* a point within equidistant from all vertices. Is this true of all regular polyedrons?

421. Similiar polyedrons. — Two polyedrons are **similar** when they have the same number of faces, similar each to each and similarly placed, and the homologous polyedral angles are equal.



SIMILAR POLYEDRONS

422. Corollary. — In two similar polyedrons:

(1) *The ratio of any two homologous edges is equal to the ratio of any other two homologous edges.*

(2) *The areas of any two homologous faces are to each other as the squares of any two homologous edges.*

(3) *The areas of the entire surfaces are to each other as the squares of any two homologous edges.*

The proof is left to the student.

EXERCISES

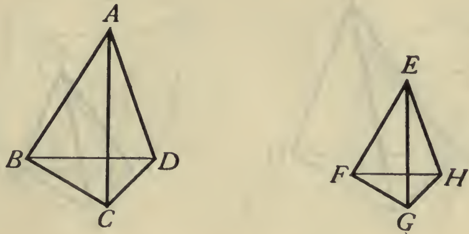
1. In two similar polyedrons, any two homologous diagonals have the same ratio as any two homologous edges.

2. Prove that any two regular tetraedrons are similar. Is this true of any two regular hexaedrons? Octaedrons? Dodecaedrons? Icosaedrons?

3. Two homologous edges of two similar polyedrons are 4 in. and 6 in., respectively. The total area of the smaller is 128 sq. in. Find the total area of the larger.

4. Verify § 422 (3), in the case of two similar rectangular parallelopipeds whose concurrent edges are 3 in., 4 in., 6 in., and 6 in., 8 in., 12 in., respectively.

423. Theorem. — *Two tetraedrons are similar when the faces including a triedral angle of one are similar, respectively, to the faces including a triedral angle of the other, and are similarly placed.*



Hypothesis. Tetraedrons $ABCD$ and $EFGH$ have
$\triangle ABC \sim \triangle EFG$, $\triangle ACD \sim \triangle EGH$, $\triangle ABD \sim \triangle EFH$.

Conclusion. $ABCD \sim EFGH$.

Proof. 1. $\triangle ABC \sim \triangle EFG$, $\triangle ACD \sim \triangle EGH$, $\triangle ABD \sim \triangle EFH$. Hyp.

2. $\therefore \dfrac{BC}{FG} = \dfrac{AC}{EG}$ and $\dfrac{CD}{GH} = \dfrac{AC}{EG}$. § 127

3. $\therefore \dfrac{BC}{FG} = \dfrac{CD}{GH}$. Ax. I

4. Similarly, $\dfrac{CD}{GH} = \dfrac{BD}{FH}$.

5. $\therefore \triangle BCD \sim \triangle FGH$. § 129

6. Now $\angle BAC = \angle FEG$, $\angle CAD = \angle GEH$, $\angle BAD = \angle FEH$. § 127

7. $\therefore$ angle $A\text{–}BCD =$ angle $E\text{–}FGH$. § 346

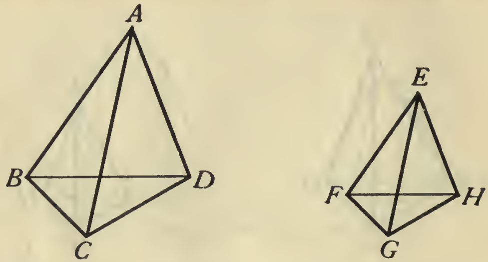
8. Similarly, the other corresponding triedral angles are equal.

9. $\therefore ABCD \sim EFGH$. § 421

EXERCISE

If a given tetraedron is cut by a plane parallel to a face, the tetraedron cut off is similar to the given one.

424. Theorem. — *Two tetraedrons are similar when a diedral angle of one is equal to a diedral angle of the other, and the including faces are similar each to each and similarly placed.*



Hypothesis. In tetraedrons $ABCD$ and $EFGH$, diedral angle $C\text{–}AB\text{–}D =$ diedral angle $G\text{–}EF\text{–}H$, $\triangle ABC \sim \triangle EFG$, $\triangle ABD \sim \triangle EFH$.

Conclusion. $ABCD \sim EFGH$.

Proof. 1. Angle $C\text{–}AB\text{–}D =$ angle $G\text{–}EF\text{–}H$, etc. Hyp.

2. $\therefore ABCD$ may be superposed upon $EFGH$ so that angle $C\text{–}AB\text{–}D$ coincides with its equal, angle $G\text{–}EF\text{–}H$, A falling at E. Test of equality

3. $\angle BAC = \angle FEG$ and $\angle BAD = \angle FEH$. § 127

4. $\therefore AC$ will fall along EG, and AD along EH. § 13

5. $\therefore \angle CAD = \angle GEH$. § 13

6. $\dfrac{AC}{EG} = \dfrac{AB}{EF}$ and $\dfrac{AD}{EH} = \dfrac{AB}{EF}$. § 127

7. $\therefore \dfrac{AC}{EG} = \dfrac{AD}{EH}$. Ax. I

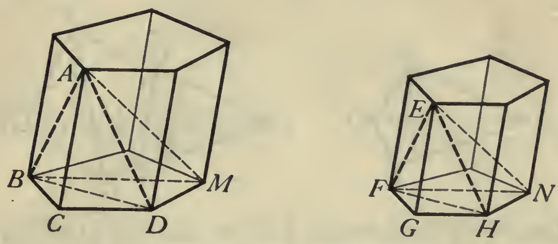
8. $\therefore \triangle ACD \sim \triangle EGH$. § 130

9. $\therefore ABCD \sim EFGH$. § 423

EXERCISE

Is this a true theorem: Tetraedron $ABCD \sim$ tetraedron $EFGH$ when $\triangle ABC \sim \triangle EFG$, angle $C\text{–}AB\text{–}D =$ angle $G\text{–}EF\text{–}H$, and angle $B\text{–}AC\text{–}D =$ angle $F\text{–}EG\text{–}H$?

425. Theorem. — *Two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each and similarly placed.*



Hypothesis. AM and EN are two similar polyedrons.

Conclusion. AM and EN may be decomposed into the same number of tetraedrons, similar each to each and similarly placed.

Proof. 1. Polyedrons AM and EN are similar. Hyp.

2. Let A and E be any two homologous vertices. Divide all faces, except those having A and E as vertices, into triangles. Pass planes through A and the vertices of the triangles in the faces of AM, and through E and the vertices of the triangles in the faces of EN.

3. Let $ABCD$ and $EFGH$ be two homologous tetraedrons formed.

4. Now face $AB \sim$ face EF. § 421

5. $\therefore \angle BCA = \angle FGE$, and $\dfrac{BC}{FG} = \dfrac{AC}{EG}$. § 127

6. $\therefore \triangle ABC \sim \triangle EFG$. § 130

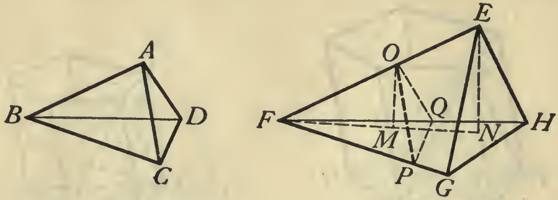
7. Similarly, $\triangle ACD \sim \triangle EGH$.

8. Also angle $B\text{–}AC\text{–}D =$ angle $F\text{–}EG\text{–}H$. § 421

9. $\therefore ABCD \sim EFGH$. § 424

10. It may now be proved that tetraedron $ABDM \sim$ tetraedron $EFHN$, etc.

426. Theorem. — *The volumes of two tetraedrons having a pair of equal triedral angles are to each other as the products of the edges including the equal triedral angles.*



Hypothesis. Tetraedrons $ABCD$ and $EFGH$ have angle $B\text{-}ACD =$ angle $F\text{-}EGH$, and their volumes are V and W, respectively.

Conclusion. $\dfrac{V}{W}=\dfrac{BA \times BC \times BD}{FE \times FG \times FH}.$

Proof. 1. Angle $B\text{-}ACD =$ angle $F\text{-}EGH$. Hyp.

2. $\therefore$ $ABCD$ may be superposed upon $EFGH$ so that angles $B\text{-}ACD$ and $F\text{-}EGH$ coincide, $ABCD$ becoming $OFPQ$. Test of equality

3. If OM and EN are $\perp$ plane FGH, $OM \parallel EN$. § 321

4. $\therefore$ OM and EN determine a plane, which contains FE and intersects plane FGH in FN. § 299, § 291, § 301

5. $\dfrac{V}{W}=\dfrac{\frac{1}{3}\,OM \times \triangle FPQ}{\frac{1}{3}\,EN \times \triangle FGH}=\dfrac{OM}{EN}\times\dfrac{\triangle FPQ}{\triangle FGH}.$ § 394

6. In $\triangle$s FMO and FNE, $\angle F$ is common, $\angle FMO = \angle FNE$, $\angle FOM = FEN$. § 26

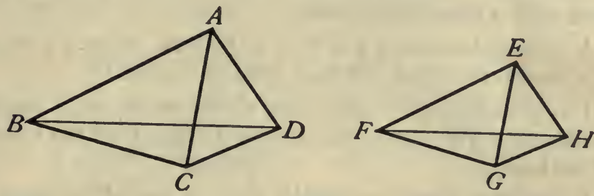
7. $\therefore \triangle FMO \sim \triangle FNE$. § 128

8. $\therefore \dfrac{OM}{EN}=\dfrac{FO}{FE}$ or $\dfrac{BA}{FE}$. § 127

9. Also $\dfrac{\triangle FPQ}{\triangle FGH}=\dfrac{FP \times FQ}{FG \times FH}$ or $\dfrac{BC \times BD}{FG \times FH}$. § 228

10. $\therefore \dfrac{V}{W}=\dfrac{BA}{FE}\times\dfrac{BC \times BD}{FG \times FH}=\dfrac{BA \times BC \times BD}{FE \times FG \times FH}.$ Ax. XII

427. Theorem. — *The volumes of two similar tetraedrons are to each other as the cubes of their homologous edges.*



Hypothesis. Tetraedrons $ABCD$ and $EFGH$ are similar, with AB and EF homologous edges, and with volumes V and W, respectively.

Conclusion. $\dfrac{V}{W}=\dfrac{\overline{AB}^3}{\overline{EF}^3}$.

Proof. 1. Tetraedron $ABCD \sim$ tetraedron $EFGH$. Hyp.

2. $\therefore$ angle $A\text{–}BCD =$ angle $E\text{–}FGH$. § 421

3. $\therefore \dfrac{V}{W}=\dfrac{AB \times AC \times AD}{EF \times EG \times EH}=\dfrac{AB}{EF}\times\dfrac{AC}{EG}\times\dfrac{AD}{EH}$. § 426

4. But $\dfrac{AB}{EF}=\dfrac{AC}{EG}=\dfrac{AD}{EH}$. § 422 (1)

5. $\therefore \dfrac{V}{W}=\dfrac{AB}{EF}\times\dfrac{AB}{EF}\times\dfrac{AB}{EF}$ or $\dfrac{\overline{AB}^3}{\overline{EF}^3}$. Ax. XII

428. Corollary. — *The volumes of any two similar polyedrons are to each other as the cubes of their homologous edges.*

The proof is left to the student.

EXERCISES

1. The edge of a cube is 16 in. Find the edge of a cube which shall have a volume twice as great as that of the given cube.

2. If the edge of a given cube is e, find the edge of a cube with n times the volume.

3. The edge of a pyramid is x. Find the edge of a similar pyramid with twice the volume.

4. The edge of a given polyedron is E. Find the edge of a similar polyedron with m times the volume.

5. A rectangular tank is 4 ft. by 6 ft. by 8 ft. What are the dimensions of a similar tank that will hold 4 times as much?

6. The volumes of two similar polyedrons are 36 cu. in. and 972 cu. in., respectively. The edge of the smaller is 3 in. Find the homologous edge of the larger.

7. A farmer had two rectangular ricks of hay of the same shape. The large rick was $1\frac{1}{2}$ times as long as the small one. He hauled away and weighed the small one, and found that it contained $2\frac{1}{2}$ tons. How many tons did the large rick contain?

8. A farmer had two similar wedge-shaped piles of corn. One was 10 ft. high and the other only 4 ft. high. The small pile was known to contain 70 bu. How many bushels in the larger pile?

9. The volume of a given octaedron is 72 cu. in. Construct a similar octaedron whose volume shall be 9 cu. in.

10. The altitude of a pyramid is 9 ft. Find where to pass two planes through it, parallel to the base, that will cut it into three equal parts.

11. From a given pyramid cut off a frustum whose volume shall be $\frac{7}{8}$ of the volume of the given pyramid.

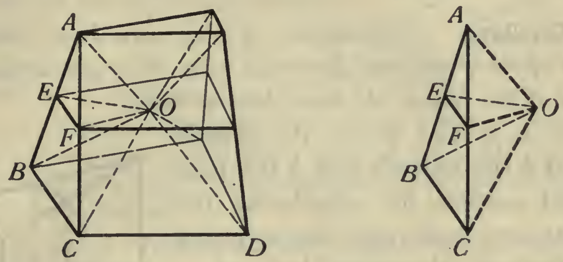
12. The volumes of two similar prisms are 64 cu. ft. and 1000 cu. ft., respectively. The total area of the smaller is 32 sq. ft. Find the total area of the larger.

13. If S and s are the total areas, and V and v the volumes of two similar polyedrons, write the equation expressing the relation between them.

429. Prismatoids. — A **prismatoid** is a polyedron bounded by two polygons in parallel planes, called the **bases**, and by lateral faces which are triangles, parallelograms, or trapezoids having one side common with one of the bases and the opposite vertex or side common with the other base.

The perpendicular distance between the planes of the bases is called the **altitude**. The section made by a plane parallel to the bases and bisecting the altitude is called the **mid-section**.

430. Theorem. — *The volume of any prismatoid is equal to one sixth of the product of its altitude by the sum of the areas of its bases and four times its mid-section.*



Hypothesis. V is the volume of a prismatoid, H the altitude, B and b areas of bases, and M area of mid-section.

Conclusion. $V = \frac{1}{6} H (B + b + 4 M)$.

Proof. 1. $V =$ volume, $H =$ altitude, B and $b =$ areas of bases, $M =$ area of mid-section. Hyp.

2. Let O be any point in mid-section. Pass a plane through O and each edge of the prismatoid, dividing it into pyramids with common vertex O.

3. The volumes of the pyramids whose bases are B and b are $\frac{1}{6} HB$ and $\frac{1}{6} Hb$, respectively. § 395

4. All pyramids having lateral faces of the prismatoid as bases may be divided into triangular pyramids, as $O\text{–}ABC$.

5. It may be proved that $\triangle ABC = 4 \times \triangle AEF$. § 228

6. $\therefore O\text{–}ABC = 4 \times O\text{–}AEF$. Ax. IV, § 394

7. But $O\text{–}AEF = A\text{–}OEF = \frac{1}{6} H \times \triangle OEF$. § 394

8. $\therefore O\text{–}ABC = \frac{1}{6} H \times (4 \times \triangle OEF)$. Ax. XII

9. In adding all such pyramids as $O\text{–}ABC$, the sum of all such triangles as $\triangle OEF$ would equal M. Ax. X

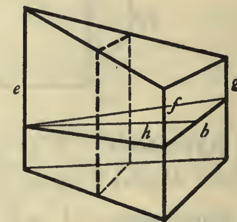
10. $\therefore$ the sum of all such pyramids $= \frac{1}{6} H \times 4 M$.

Ax. II, Factoring, Ax. XII

11. $\therefore V = \frac{1}{6} H(B + b + 4 M)$. Ax. II, Factoring

NOTE. — The formula $V = \frac{1}{6} H(B + b + 4M)$ for computing the volume of any prismatoid is very important. From it can be derived the formulæ for computing the volumes of all of the solids of elementary geometry. One illustration of this is shown in the following corollary.

431. Corollary. — *The volume of any truncated triangular prism is equal to the product of the area of a right section and one third of the sum of the three lateral edges.*



For, if e, f, and g are the lateral edges and h the altitude and b the base of a right section, by considering the edge e as one base and the opposite lateral face as the other base of a prismatoid,

$$V = \frac{1}{6} h\{0 + \frac{1}{2} b(f + g) + 4M\}.$$

But $M = \frac{1}{4} b(\frac{1}{2}[e + f] + \frac{1}{2}[e + g])$,

$$\therefore V = \frac{1}{6} h\{\frac{1}{2} b(f + g) + b(\frac{1}{2}[e + f] + \frac{1}{2}[e + g])\}$$
$$= \frac{1}{6} hb(e + f + g)$$
$$= \frac{1}{2} hb \times \frac{1}{3}(e + f + g)$$
$$= \text{area of right section} \times \frac{1}{3}(e + f + g).$$

Explain the steps of the proof.

EXERCISES

1. Show that the rule in § 370 for finding the volume of any prism follows from the prismatoid formula by making $B = b = M$.

2. Show that the rule in § 395 for finding the volume of any pyramid follows from the prismatoid formula.

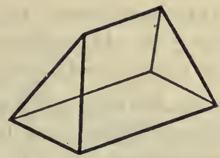
3. Show that the rule in § 397 for finding the volume of a frustum of a pyramid follows from the prismatoid formula.

SUGGESTIONS. — 1. If S and s are corresponding sides of B and b, then the corresponding side of M is $\frac{1}{2}(S + s)$.

2. Hence, show that $\dfrac{S}{\frac{1}{2}(S + s)} = \dfrac{\sqrt{B}}{\sqrt{M}}$, and $\dfrac{s}{\frac{1}{2}(S + s)} = \dfrac{\sqrt{b}}{\sqrt{M}}$.

3. By adding these equations and solving, show that

$$4M = B + b + 2\sqrt{Bb}.$$



4. A prismatoid of which one base is a rectangle, called the *base*, and the other base a line parallel to one side of this rectangle, called the *edge*, is a **wedge**.

Develop the formula for the volume of a wedge.

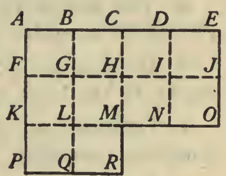
5. The base of a wedge is 5 in. by 8 in., the altitude 6 in., and the edge 4 in. Find the volume.

6. Show by use of the prismatoid formula that the volume of any truncated quadrangular prism whose opposite lateral faces are parallel equals the area of a right section, multiplied by one fourth of the sum of the four lateral edges.

7. A truncated right quadrangular prism has for base a square whose side is 3 in., and its lateral edges are 4 in., 5 in., 4 in., and 3 in., respectively. Find its volume.

8. A marble monument is in the form of a truncated right quadrangular prism whose base is a rectangle 1 ft. 6 in. by 3 ft., and lateral edges 3 ft., 3 ft., $3\frac{1}{2}$ ft., and $3\frac{1}{2}$ ft., respectively. Marble weighs 170 lb. to the cubic foot. Find the weight of the monument.

9. A bin of wheat is 8 ft. wide and 10 ft. long. The depths of the wheat at the four corners are 4 ft., 6 ft., 6 ft., and 8 ft., respectively. Find the number of bushels in the bin. (Count 4 bu. to 5 cu. ft.)



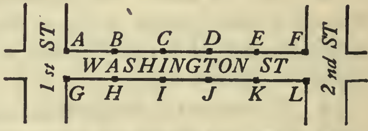
10. In finding the contents of large excavations, surveyors lay out the surface of the ground to be excavated, such as $AEOMRP$, into equal rectangles. Pegs are driven at the corners of all the rectangles, such as G, H, etc., and the depth of the cut to be made at each of these corners is found with a surveyor's level. For practical purposes, when the rectangles are small, the surface of any one rectangle may be considered plane. The whole excavation is thus divided into a number of partial volumes, each in the form of a truncated quadrangular prism. By computing the volume of each of these, and adding, the volume of the whole excavation is found. Evidently, the depth of the cut at any corner may be used 1, 2, 3, or 4 times in the computation, depending upon the number of rectangles adjoining it.

Show that the volume of the whole excavation may be found by the following rule:

"Take each corner height as many times as there are partial areas adjoining it, add them all together, and multiply by one fourth of the area of a single rectangle."

11. In the figure of Exercise 10, each rectangle is a square whose side is 50 ft. The depths of the cuts at the various corners are as follows: A, 1 ft.; B, 6 ft.; C, 3 ft.; D, 4 ft.; E, 4 ft.; F, 3 ft.; G, 5 ft.; H, 1 ft.; I, 3 ft.; J, 3 ft.; K, 2 ft.; L, 4 ft.; M, 3 ft.; N, 5 ft.; O, 4 ft.; P, 3 ft.; Q, 5 ft.; R, 6 ft. Find the number of cubic feet in the excavation. The number of cubic yards.

12. In the excavation of Washington Street, between 1st Street and 2nd Street, stakes are driven every 100 ft. on each side of the street. The width of Washington Street is 50 ft. The depths of the cuts are found to be as follows: A, 8 ft.; B, $6\frac{1}{2}$ ft.; C, 4 ft.; D, 5 ft.; E, 2 ft.; F, 1 ft.; G, 7 ft.; H, 7 ft.; I, $5\frac{1}{2}$ ft.; J, 4 ft.; K, $3\frac{1}{2}$ ft.; L, 2 ft. Find the number of cubic yards in the excavation.



MISCELLANEOUS EXERCISES

1. Two polyedrons which can be divided into the same number of tetraedrons, similar each to each and similarly placed, are similar.

2. The middle points of the edges of a regular tetraedron are the vertices of a regular octaedron.

3. If from any point within a regular tetraedron perpendiculars are drawn to the faces, their sum is equal to an altitude of the tetraedron.

4. Construct an angle equal to the plane angle of a diedral angle of a regular octaedron.

5. The volume of a regular octaedron is equal to the product of the cube of its edge and $\frac{1}{3}\sqrt{2}$.

6. If each of two polyedrons is similar to a third polyedron, they are similar to each other.

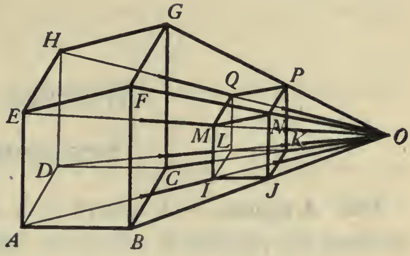
7. The homologous edges of three similar tetraedrons are 3, 4, and 5, respectively. What is the homologous edge of a similar tetraedron whose volume is equal to the sum of the volumes of the three tetraedrons?

8. If two polyedrons $ABCDE \cdots$ and $IJKLM \cdots$ are so placed that line-segments from a point O to A, B, C, etc., are divided by

points I, J, K, etc., such that

$$\frac{OA}{OI}=\frac{OB}{OJ}=\frac{OC}{OK}=\frac{OD}{OL}=\text{etc.},$$

the two polyedrons are said to be *radially placed*. Prove that polyedrons $ABCDE \cdots$ and $IJKLM \cdots$ are similar.



9. A plane passed through the point of intersection of the diagonals of any parallelopiped divides it into two equal solids.

10. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by the distance between the points of intersection of the diagonals of the bases (centers).

11. Find the area and the volume of a regular tetraedron each edge of which is 4 in.

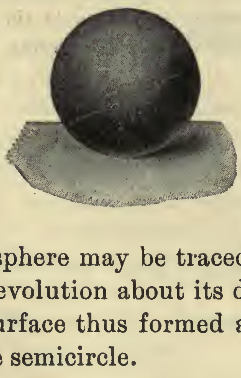
12. The total area of a given tetraedron is 256 sq. ft. Find the total area of the tetraedron cut off by a plane parallel to a face of the given tetraedron and midway between that face and the opposite vertex.

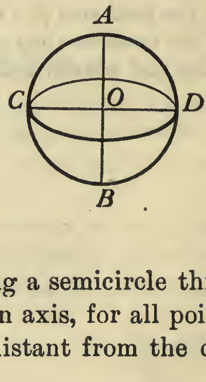
CHAPTER XVI

THE SPHERE

432. A sphere. — A **sphere** is a solid bounded by a curved surface all points of which are equally distant from a point within called the **center.** The bounding surface is called the **spherical surface**; the line-segment from the center to any point of the spherical surface, a **radius**; and a line-segment through the center and terminating in the spherical surface, a **diameter.**

As in a circle, a diameter of a sphere is equal to two radii.





A sphere may be traced by revolving a semicircle through one revolution about its diameter as an axis, for all points of the surface thus formed are equally distant from the center of the semicircle.

Thus, by revolving the semicircle ACB through one revolution about its diameter AB as an axis, the sphere with center O and diameter CD is traced.

433. Fundamental properties of spheres. — The following important properties of spheres may easily be reasoned out from the definitions above:

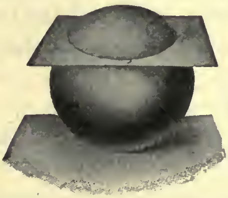
(1) *All radii of the same sphere or of equal spheres are equal.*

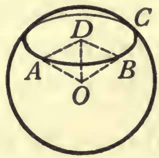
(2) *All diameters of the same sphere or of equal spheres are equal.*

(3) *Two spheres of equal radii, or of equal diameters, are equal.*

(4) *The distance of a point from the center of a sphere is equal to, greater than, or less than a radius according as it is on, without, or within the spherical surface; and conversely.*

434. Theorem. — *A section of the surface of a sphere made by a plane is a circle.*





Hypothesis. ABC is a section of the surface of a sphere with center O made by a plane.

Conclusion. ABC is a circle.

Proof. 1. ABC is a section of the surface of a sphere with center O made by a plane.

2. Draw $OD \perp$ plane of ABC, meeting it at D. Let A and B be any two points in the section. Draw OA, OB, DA, and DB.

3. Then $OA = OB$. § 433, (1)

4. $\therefore DA = DB$. § 311

5. Therefore, since A and B are any two points in section ABC, all points of the section are equidistant from D, and hence ABC is a circle. Def. ⊙

435. Circles of a sphere. — The section of the surface of a sphere made by a plane passing through the center is called a **great circle** of the sphere.

The section of a surface of a sphere made by a plane which does not pass through the center is called a **small circle** of the sphere.

The following are important properties of the circles of a sphere, and should be carefully reasoned out by the student:

(1) *All great circles of a sphere are equal.*

(2) *The plane of any great circle bisects the sphere and its surface.*

(3) *Two great circles of a sphere bisect each other.*

(4) *Two points in the surface of a sphere which are not extremities of a diameter determine a great circle of the sphere.*

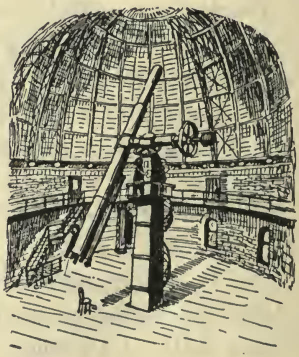
(5) *Any three points in a surface of a sphere determine a circle of the sphere.*

EXERCISES

1. What are the *meridians* on the earth's surface? How are they determined?

2. What are the lines called *parallels* on the earth's surface? Why are they so named?

3. The sky overhead appears to be a great spherical dome. Why does a telescope, when revolved in one plane about an axis, appear to describe a circle across the sky? The drawing shows the great telescope in the Yerkes Astronomical Observatory at Lake Geneva, Wis.



4. The line joining the center of a sphere and the center of a small circle of the sphere is perpendicular to the plane of the circle.

5. The radius of a sphere is 20 in. Find the radius and area of the circle made by a plane 4 in. from the center.

6. At what distance from the center of a sphere 16 in. in diameter should a plane be passed to produce a circle whose diameter is 12 in.?

7. Circles of a sphere made by planes equidistant from the center are equal.

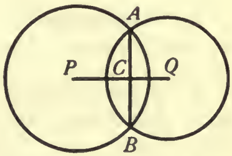
8. State and prove the converse of Exercise 7.

9. Of two circles on the same sphere, the one made by a plane more remote from the center is the smaller.

10. State and prove the converse of Exercise 9.

11. What is the largest number of points in which two circles of a sphere can intersect? Why?

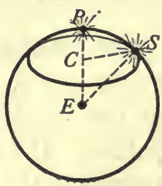
12. The intersection of the surfaces of two spheres is a circle whose center is on the line-segment joining the centers of the spheres and whose plane is perpendicular to the line-segment.



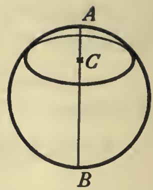
Suggestion. — Let P and Q be the centers of the two spheres, and let a plane through PQ cut the surfaces of the spheres in great circles which intersect at A and B. Prove that by revolving the figure about PQ as axis, the circles describe the surfaces of the spheres and the point A describes a circle which is their intersection.

13. The radii of two intersecting spheres are 8 in. and 10 in., respectively. The distance between their centers is 12 in. Find the radius and the area of their circle of intersection.

14. The stars appear to lie upon the surface of a sphere, called the *astronomical sphere*, with the earth at its center. Because of the daily rotation of the earth on its axis, all stars, except the Pole Star, appear to move. Since the distance from any star to the Pole Star appears to be always the same, show that the apparent daily path of a star is a circle.



Suggestion. — If P is the position of the Pole Star, E of the earth, and S of any other star, ES and $\angle SEP$ are constant. Draw $SC \perp PE$.



436. Axis and poles of a circle. — The **axis** of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle.

The **poles** of a circle of a sphere are the extremities of the axis of the circle.

Thus, if AB is the diameter of a sphere which is perpendicular to the plane of a circle of the sphere at C, AB is the axis of the circle and A and B are its poles.

It follows from the proof in § 434 that:

The axis of a circle of a sphere passes through the center of the circle.

EXERCISES

1. Considering the earth as a sphere, the North Pole and South Pole are really poles of what circles?

2. A great circle of a sphere which passes through one pole of a circle must pass through the other pole also.

3. Circles of a sphere made by parallel planes have the same axis and the same poles.

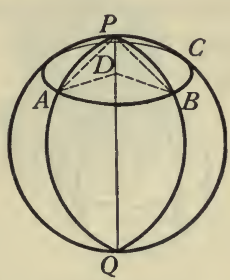
4. A given point on the surface of a sphere is the pole of one and only one great circle.

5. The line-segment which is perpendicular to the plane of a circle of a sphere at the center of the circle and terminates in the surface of the sphere is the axis of the circle.

Suggestion. — This may be proved if it is first shown that the given line-segment coincides with the axis. Hence begin by drawing the axis.

437. Distance on the surface of a sphere. — The **distance** between any two points on the surface of a sphere is the length of the smaller arc of the great circle passing through the points. If the two arcs are equal, either may be taken to represent the distance.

438. Theorem. — *Either pole of a circle of a sphere is equidistant from all points on the circle.*



Hypothesis. A and B are any two points on circle ABC, P and Q are the poles of circle ABC, and $\widehat{AP}$, $\widehat{BP}$, $\widehat{AQ}$, and $\widehat{BQ}$ are arcs of great circles.

Conclusion. $\widehat{AP} = \widehat{BP}$ and $\widehat{AQ} = \widehat{BQ}$.

Proof. 1. A and B are any two points on circle ABC of which P and Q are the poles. Hyp.

2. Let axis PQ meet the plane of ABC at D, the center of the circle. Draw AD, BD, and chords AP and BP.

3. $PQ \perp$ plane of ABC. § 436

4. $AD = BD$. § 151, (2)

5. $\therefore AP = BP$. § 310

6. $\therefore \widehat{AP} = \widehat{BP}$. § 151, (7)

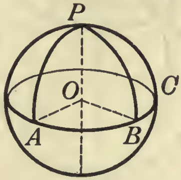
7. Similarly it may be proved that $\widehat{AQ} = \widehat{BQ}$.

439. Polar distance. — The **polar distance** of a circle of a sphere is the distance from any point of the circle to the nearer pole. Thus, in the figure of § 438, $\widehat{AP}$ is the polar distance of circle ABC.

440. Corollary. — *The polar distance of a great circle is a quadrant.*

The proof is left to the student.

441. Theorem. — *If a point on the surface of a sphere is at a quadrant's distance from each of two other points on the surface which are not extremities of a diameter, it is a pole of the great circle determined by those points.*



Hypothesis. P, A, and B are points on the surface of the sphere with center O; $\widehat{AP}$ and $\widehat{BP}$ are quadrants; ABC is the great circle determined by A and B.

Conclusion. P is the pole of ABC.

Suggestions. It will follow that P is the pole of ABC if it is proved that $PO \perp$ plane of ABC. Why? The latter will follow if what is first proved? Hence begin by drawing AO, BO, and PO, and proving $\angle\ AOP$ and $\angle\ BOP$ right angles.

EXERCISES

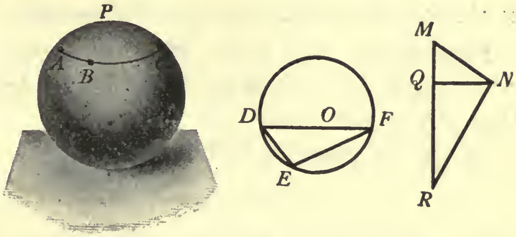
1. Prove that if the distance between two points on a sphere is a quadrant, each point is the pole of one great circle through the other.

2. Prove that if A is a point on the surface of a *material* sphere, globe, or spherical blackboard, a circle BCD with A as its pole may be drawn on the surface by holding one end of a piece of cord at A, inserting a pencil or crayon in a loop at the other end of the cord, and marking along the surface while the cord is kept stretched. Compasses may be used instead of the cord.

3. Prove that the diameter of a given *material* sphere may be obtained with straightedge and compasses as follows:

With any point P of the surface as pole, draw any circle ABC.

With compasses, measure the chords joining any three points A, B, and C of this circle, and construct a plane triangle DEF with its sides equal to these chords.



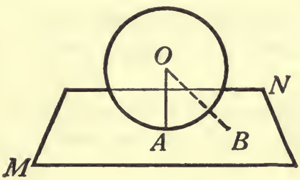
Circumscribe a circle about $\triangle DEF$, and let O be its center.

Construct right $\triangle MNQ$ with hypotenuse $MN =$ chord PC, and leg $QN =$ radius OF, and complete right $\triangle MNR$.

Then MR is the diameter required.

442. Lines and planes tangent to a sphere. — A straight line or a plane is **tangent to a sphere** if it has in common with the surface of the sphere one and only one point, called the **point of contact**.

443. Theorem. — *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



Hypothesis. O is the center of a sphere, OA is a radius, and plane MN is perpendicular to OA at A.

Conclusion. MN is tangent to the sphere with center O.

Suggestions. It may be proved that MN is tangent to the sphere if it is shown that any point other than A of plane MN, such as B, is outside of the sphere. Why? This will follow if it is first proved that $OB > OA$. Why?

Write the proof in full.

444. Theorem. — *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

Suggestions. Let plane MN be tangent at A to a sphere with center O. Let B be any point of MN except A. Show that $OB > OA$, etc.

Write the proof in full.

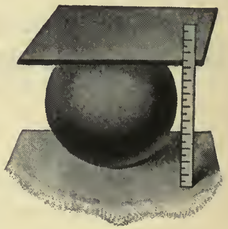
EXERCISES

1. A perpendicular to a tangent plane at the point of tangency passes through the center of the sphere.

2. A line perpendicular to a tangent plane from the center of a sphere passes through the point of contact.

3. Two planes tangent to a sphere at the extremities of the same diameter are parallel.

4. Show that the diameter of any solid spherical ball may be found by laying the ball upon a table, resting a board upon it in a horizontal position, and measuring the perpendicular distance from the edge of the board to the table.



SUGGESTION. — Draw a diameter perpendicular to the table top.

5. A straight line perpendicular to the radius of a sphere at its extremity is tangent to the sphere.

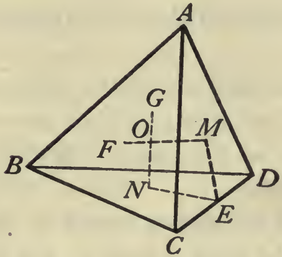
6. A straight line tangent to a sphere is perpendicular to the radius drawn to the point of contact.

7. All straight lines tangent to a sphere at the same point lie in the plane which is tangent to the sphere at that point.

445. Circumscribed and inscribed spheres of a polyedron. — A sphere is **circumscribed about a polyedron** when all of the vertices of the polyedron lie in the surface of the sphere. The polyedron is said to be **inscribed in the sphere.**

A sphere is **inscribed in a polyedron** when all of the faces of the polyedron are tangent to the sphere. The polyedron is said to be **circumscribed about the sphere.**

446. Theorem. — *A sphere can be circumscribed about any tetraedron.*



Hypothesis. $ABCD$ is any tetraedron.

Conclusion. A sphere can be circumscribed about $ABCD$.

Proof. 1. $ABCD$ is any tetraedron. Hyp.

2. Let M and N be the centers of the circumscribed circles of $\triangle ACD$ and $\triangle BCD$, respectively.

3. Then, M and N lie in perpendicular bisectors ME and NE of CD. § 105

4. $\therefore$ plane $MEN \perp$ line CD. § 303

5. $\therefore$ plane $MEN \perp$ planes ACD and BCD. § 331

6. Let MF be $\perp$ plane ACD and NG be $\perp$ plane BCD.

7. Then MF and NG both lie in plane MEN. § 333

8. $\therefore$ MF and NG intersect at some point O. § 55

9. O is equidistant from A, C, and D, and also equidistant from B, C, and D. § 310

10. $\therefore$ O is equidistant from A, B, C, and D. Ax. I

11. $\therefore$ the surface of a sphere with center O and radius OA will pass through A, B, C, and D. Def. sphere

12. $\therefore$ this sphere is circumscribed about $ABCD$. § 445

EXERCISES

1. A sphere can be circumscribed about any rectangular parallelopiped.

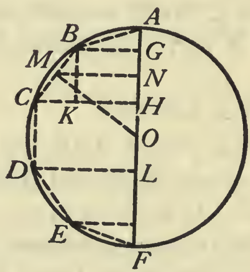
2. A sphere can be inscribed in any cube.

447. Principles of limits. —*If one half of a regular polygon having an even number of sides is inscribed in a semicircle, and the whole figure is revolved about the diameter of the semicircle as axis, then:*

(1) *the area of the surface of the sphere traced is the limit approached by the area of the surface traced by the perimeter of the half polygon as the number of sides of the polygon is indefinitely increased;*

(2) *the volume of the sphere traced is the limit approached by the volume of the solid traced by the half polygon as the number of sides of the polygon is indefinitely increased.*

448. Theorem. — *The area of the surface of a sphere is equal to four times the area of a great circle of the sphere.*



Hypothesis. S is the area of the surface and R the radius of the sphere with center O, traced by revolving semicircle ACF about diameter AF as axis.

Conclusion. $S = 4\,\pi R^2$.

Proof. 1. $S =$ area of surface, $R =$ radius, O is center of sphere traced by semicircle ACF revolved about diameter AF as axis. Hyp.

2. Inscribe in the semicircle one half of a regular polygon of an even number of sides, $ABC \cdots$. Draw $OM \perp$ any

side, as BC. From B, M, and C draw $\perp$s BG, etc., to AF. And draw $BK \perp CH$.

3. In revolving trapezoid $BCHG$ about AF as axis, BC will trace the lateral surface of a frustum of a right circular cone whose area equals $\pi \times BC \times (BG + CH)$. § 412

4. But M is the middle point of BC. § 156

5. BG, MN, and CH are parallel. § 41

6. $\therefore MN = \frac{1}{2}(BG + CH)$. § 100

7. $\therefore$ lateral area of frustum traced by $BC = 2\pi \times BC \times MN$. Ax. XII

8. It may be proved that $\triangle CKB \sim \triangle MNO$. (Proof left to student.)

9. $\therefore \dfrac{OM}{BC} = \dfrac{MN}{BK}$. Def. sim. $\triangle$s

10. But $BK = GH$. § 85

11. $\therefore \dfrac{OM}{BC} = \dfrac{MN}{GH}$. Ax. XII

12. $\therefore OM \times GH = BC \times MN$. Ax. IV

13. $\therefore$ area of surface traced by $BC = 2\pi \times OM \times GH$. Ax. XII

14. Also it can be proved that
area of surface traced by $AB = 2\pi \times OM \times AG$,
area of surface traced by $CD = 2\pi \times OM \times HL$, etc.

15. $\therefore$ if s represents the area of the surface traced by $ABC \cdots$, $s = 2\pi \times OM \times AG + 2\pi \times OM \times GH + 2\pi \times OM \times HL +$ etc. Ax. II

16. $\therefore s = 2\pi \times OM \times (AG + GH + HL +$ etc.$)$. Fact.

17. $\therefore s = 2\pi \times OM \times 2R$. Ax. XII

18. If the number of sides of the polygon is indefinitely increased, $s \doteq S$ and $OM \doteq R$. § 447, § 274

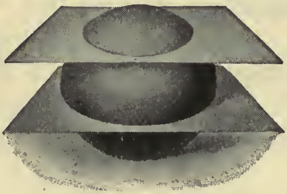
19. $\therefore 2\pi \times OM \times 2R \doteq 2\pi \times R \times 2R$ or $4\pi R^2$. § 275, (2)

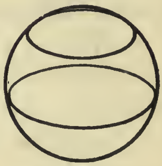
20. $\therefore S = 4\pi R^2$. § 275, (1)

449. Corollary 1. — *The areas of the surfaces of two spheres are to each other as the squares of their radii or of their diameters.*

The proof is left to the student.

450. Zones. — A **zone** is that portion of the surface of a sphere which lies between two parallel planes.





ZONE

The sections of the surface made by the planes are the **bases** of the zone. The perpendicular distance between the planes is the **altitude** of the zone.

If one of the planes is tangent to the sphere, the zone is called a **zone of one base.**

When a semicircle is revolved about the diameter as an axis, the surface traced by any arc of the semicircle is a zone.

Thus, in the figure of § 448, $\overset{\frown}{BD}$ traces a zone, and GL is the altitude.

451. Corollary 2. — *The area of a zone of a sphere is equal to the product of its altitude and the circumference of a great circle.*

For, the method of proof in § 448 applies equally to the zone traced by an arc of the revolving semicircle. The sides of the inscribed polygon which are inscribed in the given arc trace surfaces whose areas are expressed as in step 14 of § 448. By adding these areas and finding the limit of their sum, the equation $S = 2\pi RH$ is obtained, where S is the area and H the altitude of the zone, and R the radius of the sphere.

EXERCISES

1. Considering the entire surface of a sphere as a zone, deduce the formula for computing the area of the surface of a sphere from § 451.

2. Show that the area of the surface of a sphere is found by the formula $S = \pi D^2$, where D is the diameter.

3. What is the area of the surface of a sphere whose diameter is 10 in.?

4. The area of the surface of a sphere is equal to the product of the diameter and the circumference of a great circle.

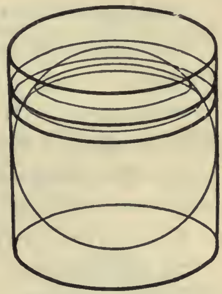
5. The diameter of a given sphere is 12 in. If parallel planes divide the diameter into four equal parts, find the area of each of the four zones thus cut off.

6. Zones on the same sphere or on equal spheres are to each other as their altitudes.

7. The diameter of a sphere is 8 in., and a plane is passed through the sphere 1 in. from the center. Compare the two parts into which the surface of the sphere is divided.

8. Find the diameter of a sphere the area of whose surface is 96 sq. ft.

9. The figure represents a sphere inscribed in a right circular cylinder, the bases of the cylinder being tangent to the sphere and the lateral surface of the cylinder touching the surface of the sphere in a circle of contact. Corresponding belts are formed on the two surfaces, included between two planes each parallel to the bases of the cylinder. Prove that the areas of the two belts are equal. Show that the entire area of the surface of the sphere is equal to the lateral area of the cylinder.



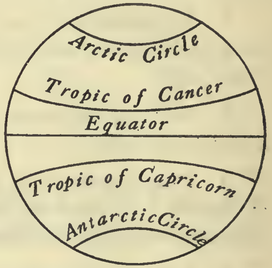
10. The area of the surface of a sphere is equal to two thirds of the total area of the surface of the circumscribed right circular cylinder.

NOTE. — The discovery of the relation in Exercise 10 between the areas of the surfaces of a sphere and its circumscribed cylinder, and of a like relation between the volumes of these solids, was first made by Archimedes. He was so pleased with these discoveries that he asked that the figure of a sphere and its circumscribed cylinder be inscribed on his tomb. This request was carried out by his friend Marcellus.

11. Prove that a zone of one base has the same area as a circle whose radius is the chord of the generating arc of the zone.

Suggestion. — This necessitates showing that the square of the chord is equal to 2 times the product of the radius and the altitude of the zone. Why?

12. Taking the earth as a sphere and the radius as 4000 miles, find the area of the entire surface of the earth. The area of the United States is 3,624,122 sq. mi. What part of the earth's surface is occupied by the United States?



13. The North Frigid Zone is that part of the earth's surface lying north of the Arctic Circle and extending to the North Pole. Its altitude is approximately 330 mi. Find its area.

What part of the earth's surface lies in the two frigid zones?

14. Find the diameter of the Arctic Circle.

15. The North Temperate Zone is that part of the earth's surface lying between the Tropic of Cancer and the Arctic Circle. Its altitude is approximately 2165 mi. Find its area.

What part of the earth's surface lies in the two temperate zones?

16. Prove that one half of the earth's surface lies within 30° of the equator.

Suggestion. — What is the relation between the sides of a right triangle in which one of the acute angles is 30°?

17. How many miles above the surface of the earth is that point from which one fourth of the earth's surface may be seen?

Suggestion. — Apply the same principles as in Exercise 16.

18. An observer stands d feet from the center of a sphere whose diameter is d feet. What part of the surface of the sphere can be seen?

19. Find the total area of a hemispherical bowl 1 in. thick whose external diameter is 12 in.

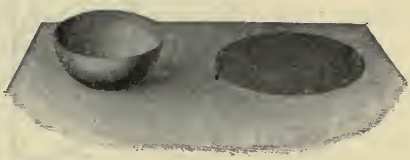
Suggestion. — We are to find the surfaces of two hemispheres and a ring (the rim).

20. The surface of a tiled dome, in the form of a hemispherical surface whose diameter is 24 ft., is made of colored tiles each 1 in. square. How many tiles are required to make it?

21. In a zone of one base, the diameter of the base is 16 in. and the altitude 4 in. Find the radius of the sphere.

SUGGESTION. — If x is the radius of the sphere, $4(2x - 4) = 64$.

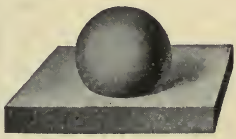
22. Vessels whose surfaces are in the form of zones of one base sometimes are made from sheet metal, by cutting out a circular blank from a flat sheet of the metal, and pressing it into the required form in a die, as described in the exercises on the cylinder and the cone. Find the diameter of the blank required to make a hemispherical brass bowl whose diameter is 8 in.



23. The lid to a silver-plated dish is in the form of a zone of one base, with a flange. The depth of the lid is $1\frac{1}{2}$ in., the width of the flange 1 in., and the outer diameter of the flange 9 in. Find the diameter of the blank from which it is made.

24. Hollow metal balls, used as ornaments, casters, anti-friction balls, etc., are made from circular blanks cut from sheet metal, by use of a die. Find the diameter of the blank required to make a hollow metal ball $\frac{1}{2}$ in. in diameter.

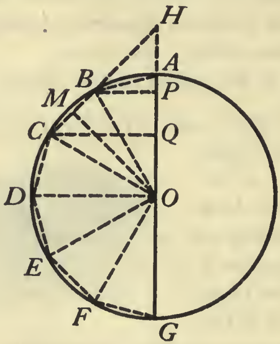
25. A method of testing the hardness of metals consists in partly forcing a hardened steel ball into the sample being tested. The *hardness numeral* H is computed by the formula $H = \frac{K}{y}$, where K is pressure on ball in kilograms, and y is area of surface of depression in sample in square millimeters.



Show that $y = 2\pi r(r - \sqrt{r^2 - R^2})$, where $r =$ radius of ball and $R =$ radius of depression in sample.

Substitute this value of y, and deduce the formula for finding H in terms of K, r, and R.

452. Theorem. — *The volume of a sphere is equal to the product of the area of its surface and one third of the radius.*



Hypothesis. $V =$ the volume, $S =$ the area of the surface, and $R =$ the radius of a sphere.

Conclusion. $V = \frac{1}{3} RS.$

Proof. 1. V, S, and R are the volume, area of surface, and radius, respectively, of sphere. Hyp.

2. Let AG be the diameter of the semicircle by revolving which the sphere is traced. Inscribe in the semicircle one half of a regular polygon of an even number of sides, $ABCD \cdots$. Produce any side, say BC, to meet AG produced at some point H. Draw OB and OC. Draw $OM \perp BC$. Draw $BP \perp AG$ and $CQ \perp AG$.

3. Vol. traced by OCH

$= $ vol. traced by $OCQ +$ vol. traced by CQH

$= \frac{1}{3}\pi \times \overline{CQ}^2 \times OQ + \frac{1}{3}\pi \times \overline{CQ}^2 \times QH$ § 414

$= \frac{1}{3}\pi \times \overline{CQ}^2 \times (OQ + QH)$ Factoring

$= \frac{1}{3}\pi \times CQ \times CQ \times OH.$ Ax. XII

4. But it may be proved that $\triangle OMH \sim \triangle CQH$. (Proof left to student.)

5. $\therefore \dfrac{CQ}{OM} = \dfrac{CH}{OH}.$ Def. sim. $\triangle$s

6. $\therefore CQ \times OH = OM \times CH$. Ax. IV

7. $\therefore$ vol. traced by $OCH = \frac{1}{3}\pi \times CQ \times OM \times CH$. Ax. XII

8. But $\pi \times CQ \times CH =$ area of surface traced by CH. § 410

9. $\therefore$ vol. traced by $OCH = \frac{1}{3} OM \times$ area traced by CH. Ax. XII

10. Similarly, vol. traced by $OBH = \frac{1}{3} OM \times$ area traced by BH.

11. $\therefore$ vol. traced by OBC
$= \frac{1}{3} OM \times$ area traced by $CH - \frac{1}{3} OM \times$ area traced by BH
$= \frac{1}{3} OM \times$ area traced by BC. Factoring, Ax. XII

12. Similarly, vol. traced by $OAB = \frac{1}{3} OM \times$ area traced by AB, vol. traced by $OCD = \frac{1}{3} OM \times$ area traced by CD, etc.

13. $\therefore$ if v is volume traced by half polygon $ABCD \cdots$, and s is the area traced by its perimeter,
$v = \frac{1}{3} OM \times$ (area traced by $AB +$ area traced by $BC +$ etc.)
$= \frac{1}{3} OM \times s$. Ax. II, Factoring, Ax. XII

14. If the number of sides of the polygon is indefinitely increased, $v \doteq V$, $s \doteq S$, and $OM \doteq R$. § 447, § 274

15. $\therefore \frac{1}{3} OM \times s \doteq \frac{1}{3} RS$. § 275 (2), § 408 (2)

16. $\therefore V = \frac{1}{3} RS$. § 275 (1)

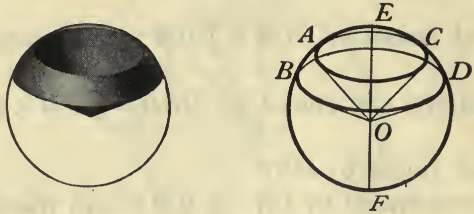
453. Corollary 1. — *The volume of a sphere equals* $\frac{4}{3}\pi R^3$, *or* $\frac{1}{6}\pi D^3$, *where R is the radius and D the diameter.*

The proof is left to the student.

454. Corollary 2. — *The volumes of any two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

The proof is left to the student.

455. A spherical sector. — When a sphere is traced by revolving a semicircle about its diameter as an axis, that portion of the sphere which is traced by a sector of the circle is called a **spherical sector.**

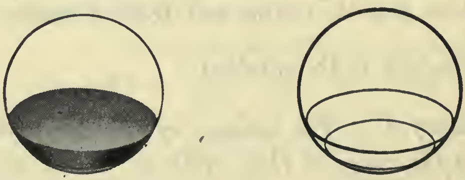


Thus, if the sector AOB of a circle is revolved about the diameter EF as an axis, it traces the spherical sector $AB–O–CD$.

The zone traced by the arc of the sector of a circle is called the **base** of the spherical sector.

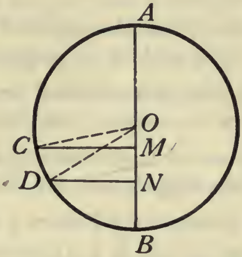
456. Corollary 3. — *The volume of a spherical sector is equal to the product of the area of its base and one third of the radius of the sphere.* (Apply § 452.)

457. A spherical segment. — A **spherical segment** is the portion of a sphere included between two parallel planes. The sections of the sphere made by the planes are the **bases**, and the perpendicular distance between the planes is the **altitude.**



When one of the bounding planes is tangent to the sphere, the segment is called a **spherical segment of one base.**

458. Theorem. — *The volume of a spherical segment of which the altitude is h and of which the radii of the bases are a and b, respectively, equals* $\frac{1}{2}\pi h(a^2+b^2)+\frac{1}{6}\pi h^3$.



Hypothesis. The volume of a spherical segment is V, the altitude is h, and the radii of its bases are a and b.

Conclusion. $V=\frac{1}{2}\pi h\,(a^2+b^2)+\frac{1}{6}\pi h^3$.

Proof. 1. Vol. sph. seg. = V, alt. = h, etc. Hyp.

2. Let the spherical segment be traced by revolving $DNMC$, one half of a segment of a circle, about diameter AB as axis. Then $OB=R$, $MN=h$, $CM=a$, and $DN=b$. Draw OC and OD. Let $ON=x$ and $OM=y$.

3. Then $V=$ vol. traced by $OCD+$ vol. traced by $ODN-$ vol. traced by $OCM=\frac{2}{3}\pi R^2h+\frac{1}{3}\pi b^2x-\frac{1}{3}\pi a^2y$. § 456, § 414

4. But $h=x-y$, $b^2=R^2-x^2$, $a^2=R^2-y^2$. § 196

5. $\therefore V=\frac{1}{3}\pi\{2R^2(x-y)+(R^2-x^2)x-(R^2-y^2)y\}$
$=\frac{1}{3}\pi(x-y)\{3R^2-(x^2+xy+y^2)\}$
$=\frac{1}{3}\pi h\{3R^2-(x^2+xy+y^2)\}$. Ax. XII, Fact.

6. But $x^2-2xy+y^2=h^2$. Ax. VI

7. Subtracting this from identity $3x^2+3y^2=3x^2+3y^2$,
$$2x^2+2xy+2y^2=3x^2+3y^2-h^2.$$

8. $\therefore x^2+xy+y^2=\frac{3}{2}(x^2+y^2)-\frac{1}{2}h^2$ Ax. V
$=\frac{3}{2}(R^2-b^2+R^2-a^2)-\frac{1}{2}h^2$ Ax. XII
$=3R^2-\frac{3}{2}(a^2+b^2)-\frac{1}{2}h^2$.

9. $\therefore V=\frac{1}{2}\pi h\,(a^2+b^2)+\frac{1}{6}\pi h^3$. Ax. XII

EXERCISES

1. Find the volume of a sphere whose diameter is 16 in.

2. Find the volume of a spherical sector if its altitude is 3 in. and the radius of the sphere is 8 in.

3. Find the volume of a spherical segment if its altitude is 4 in. and the radii of its bases are 12 in. and 16 in., respectively.

4. If V is the volume of a spherical sector and H is the altitude of its base, and R is the radius of the sphere, prove that $V = \frac{2}{3}\pi R^2H$.

5. Prove that the volume of a spherical segment of one base equals $\pi h^2(R - \frac{1}{3}h)$, where h is the altitude and R is the radius of the sphere.

SUGGESTION. — In the formula of § 458, let $b = 0$, and apply § 200.

6. Find the ratio of the volume of a sphere to that of a circumscribed cube.

7. Find the ratio of the volume of a sphere to that of an inscribed cube.

8. Prove that the volume of any sphere is equal to two thirds of the volume of the circumscribed right circular cylinder.

9. A sphere whose diameter is D is transformed into a cylinder of revolution having the same diameter. Find the height of the cylinder.

10. Show that the formula for computing the volume of a sphere may be obtained from the formula for computing the volume of a spherical sector.

11. Show that the formula for computing the volume of a sphere may be obtained from the formula for computing the volume of a spherical segment.

12. Prove that if a sphere and a cube have equal areas, the sphere has the greater volume.

NOTE. — It is interesting to note that of all solids having equal areas, the sphere has the greatest volume.

13. Two spheres of lead, of radii 2 in. and 4 in., respectively, are melted and recast into a solid cylinder of revolution whose altitude is 6 in. Show that the total surface is unchanged in amount.

14. A cubic foot of ivory weighs 114 lb. What is the weight of an ivory billiard ball 2 in. in diameter?

15. A hollow spherical steel shell is 1 in. thick, and its outer diameter is 10 in. Allowing 490 lb. to a cubic foot, find its weight.

16. A washbasin is in the form of a spherical segment of one base. Its depth is $7\frac{1}{2}$ in. and the distance across the top is 16 in. Find how many gallons of water it will hold, allowing $7\frac{1}{2}$ gal. to a cubic foot.

17. Through the center of a 10-in. sphere a hole was bored, taking off one fourth of the surface of the sphere from each end. Find the volume of the ring left.

18. A boiler is made in the form of a 4-ft. cylinder 2 ft. in diameter, with hemispherical ends. How many gallons will it hold?

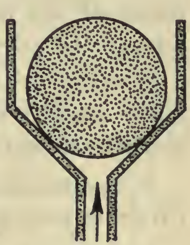
19. A sphere of lead 5 inches in diameter is formed into a tube $2\frac{1}{2}$ ft. long whose internal diameter is 1 in. Find the thickness of the tube.

20. A hemispherical boiler whose diameter is 4 ft. contains water to a depth of 20 in. How many gallons of water does it contain?

21. Among the rules of thumb used by men in practical work is the following: The weight of a cast-iron ball (in pounds) equals the cube of the diameter times .1377. Allowing 450 lb. to a cubic foot of cast iron, test the accuracy of this rule.

22. Which is the better bargain, oranges 3 in. in diameter at 30 ¢ a dozen, or oranges $3\frac{1}{2}$ in. in diameter at 40 ¢ a dozen?

23. The figure represents a cross section of a ball safety valve. The pressure of the steam from below lifts the ball, and allows the steam to escape. The diameter of the pipe below the ball is $\frac{1}{8}$ in., and the diameter of the ball is 1 in. If the ball weighs .28 lb. to the cubic inch, what is the pressure in pounds per square inch of the steam in the pipe below the ball when it lifts it and escapes?

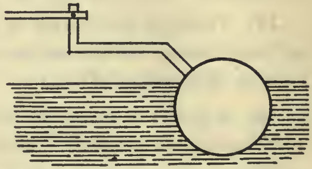


24. A haystack is approximately in the form of a cylinder 16 ft. in diameter and 8 ft. high, surmounted by a hemisphere. Allowing 512 cu. ft. to the ton, find the weight of the stack.

25. A building lot which is being leveled off has a mound on it approximately in the form of a spherical segment of one base. It is 40 ft. wide at the base and 10 ft. deep at the highest point. How many cubic yards of earth must be removed in cutting it down to the level of the surrounding ground?

26. When an object floats in water, or is immersed in it, the object is buoyed up by the water with a force equal to the weight of the water displaced by it.

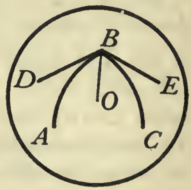
A ball floats half submerged in water. Its diameter is 8 in. Find its weight. (Water weighs 62.5 lb. per cubic foot.)



27. A float in a water tank consists of a hollow copper ball which closes the valve that lets the water enter the tank, by means of a lever to which it is attached, when the water buoys it up to a certain point. The valve shuts off the entering water when the ball is submerged to three fourths of its depth. If the diameter of the float is 4 in., what is the upward pressure of the water against it?

FIGURES ON THE SURFACE OF A SPHERE

459. Spherical angles. — A **spherical angle** is the figure formed by two arcs of great circles drawn from the same point on the surface of a sphere. The point from which the arcs are drawn is the **vertex** of the angle, and the arcs are the **sides** of the angle.



A spherical angle is *measured* by the plane angle which is formed by the tangents to its sides at the vertex.

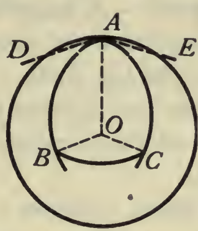
Thus, spherical angle ABC is measured by $\angle DBE$, where BD and BE are tangents to $\widehat{BA}$ and $\widehat{BC}$ respectively.

A spherical angle of which the tangents to the sides at the vertex form a right angle is called a **right spherical angle.**

460. Perpendicular arcs. — The sides of a right spherical angle are **perpendicular arcs.**

Thus, if $\widehat{BA}$ and $\widehat{BC}$ in the figure of § 459 form a right spherical angle, they are called perpendicular arcs.

461. Theorem. — *A spherical angle has the same measure as the arc of the great circle having the vertex of the angle as pole and included between the sides of the angle, produced if necessary.*



Hypothesis. $\angle BAC$ is a spherical angle on the sphere with center O; $\overset{\frown}{BC}$ is the arc of a great circle having A as its pole.

Conclusion. $\angle BAC$ has the same measure as $\overset{\frown}{BC}$.

Proof. 1. $\angle BAC$ is spherical angle; $\overset{\frown}{BC}$ is arc of great circle having A as pole. Hyp.

2. Draw radii OA, OB, OC. Draw AD tangent to $\overset{\frown}{AB}$ and AE tangent to $\overset{\frown}{AC}$.

3. $OA \perp$ plane BOC.	§ 436
4. $\therefore OB \perp OA$ and $OC \perp OA$.	§ 302
5. $AD \perp OA$ and $AE \perp OA$.	§ 164
6. $\therefore AD \parallel OB$ and $AE \parallel OC$.	§ 41
7. $\therefore \angle DAE = \angle BOC$.	§ 317
8. But $\angle BAC$ has the same measure as $\angle DAE$.	§ 459
9. And $\overset{\frown}{BC}$ has the same measure as $\angle BOC$.	§ 173
10. $\therefore \angle BAC$ has the same measure as $\overset{\frown}{BC}$.	Ax. I

462. Corollary 1. — *A spherical angle has the same measure as the diedral angle formed by the planes of its sides.*

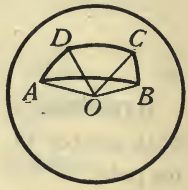
The proof is left to the student.

463. Corollary 2. — *An arc of a great circle drawn through the pole of another great circle is perpendicular to that circle.*

The proof is left to the student.

464. Spherical polygons. — A **spherical polygon** is the figure formed by three or more arcs of great circles which completely inclose a portion of the surface of a sphere.

The arcs are the **sides**, the spherical angles formed by the sides are the **angles**, and the vertices of the angles are the **vertices** of the polygon.



A **diagonal** of a spherical polygon is the great circle arc which joins any two non-adjacent vertices.

A spherical polygon each of whose angles is less than 180° is **convex.** Only convex spherical polygons are considered here.

A **spherical triangle** is a spherical polygon of three sides.

The terms **right triangle, isosceles, equilateral,** etc., are applied to spherical triangles with the same meanings as when applied to plane triangles.

465. Relation of spherical polygons to polyedral angles. — The planes of the sides of a spherical polygon form a polyedral angle whose vertex is the center of the sphere.

Thus, in the figure of § 464, the planes of the sides of polygon $ABCD$ form the polyedral angle $O\text{–}ABCD$.

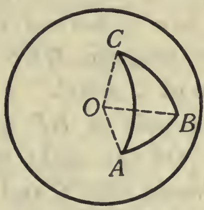
It is evident from the preceding sections that:

(1) *Each face angle of the polyedral angle has the same measure as the corresponding side of the spherical polygon.*

(2) *Each diedral angle of the polyedral angle has the same measure as the corresponding angle of the spherical polygon.*

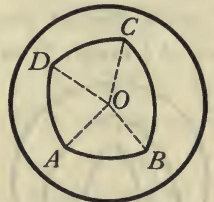
Hence, from any property of polyedral angles an analogous property of spherical polygons may be inferred.

466. Theorem. — *Any side of a spherical triangle is less than the sum of the other two sides.*



The proof is left to the student. See § 342 and § 425 (1). Write the proof in full.

467. Theorem. — *The sum of the sides of any spherical polygon is less than* 360°.



The proof is left to the student. Write the proof in full.

EXERCISES

1. The perimeter of any spherical polygon is less than a great circle.

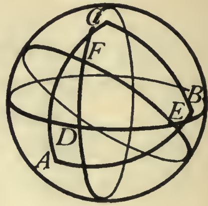
2. Any side of a spherical polygon is less than the sum of the others.

3. No side of a convex spherical polygon is as long as a semicircle.

4. If two sides of a spherical triangle are quadrants, the angles opposite them are right angles.

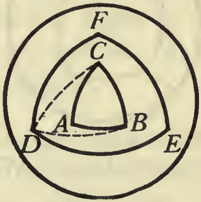
5. If two sides of a spherical triangle are 80° and 90°, respectively, between what two values does the third side lie?

468. Polar triangles. — If $\triangle ABC$ is any spherical triangle, and if the three great circles are drawn of which A, B, and C, respectively, are poles, these circles evidently form eight new spherical triangles. Of these, one triangle DEF is called the **polar triangle** of $\triangle ABC$. A is the pole of EF, B is the pole of DF, and C is the pole of DE. Of the eight triangles, $\triangle DEF$ is so chosen that C and F are on the same side of AB, A and D on the same side of BC, and B and E on the same side of AC.



NOTE. — The properties of polar triangles were discovered by Girard, a Dutch mathematician, about 1626 A.D. They were discovered independently about the same time by Snell, a prodigy, who at the age of twelve was familiar with the standard works on higher mathematics of that time.

469. Theorem. — *If one spherical triangle is the polar of a second, the second spherical triangle is the polar of the first.*



Hypothesis. $\triangle DEF$ is the polar of $\triangle ABC$.

Conclusion. $\triangle ABC$ is the polar of $\triangle DEF$.

Proof. 1. $\triangle DEF$ is the polar of $\triangle ABC$. Hyp.

2. Draw $\overset{\frown}{BD}$ and $\overset{\frown}{CD}$ of great circles.

3. Since B is the pole of $\overset{\frown}{DF}$, $\overset{\frown}{BD}$ is a quadrant. § 440

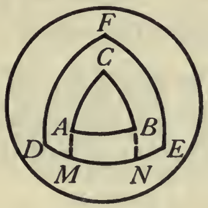
4. Since C is the pole of $\overset{\frown}{DE}$, $\overset{\frown}{CD}$ is a quadrant. § 440

5. $\therefore$ D is the pole of $\overset{\frown}{BC}$. § 441

6. Similarly, E is the pole of $\overset{\frown}{AC}$ and F the pole of $\overset{\frown}{AB}$.

7. $\therefore$ $\triangle ABC$ is the polar of $\triangle DEF$. § 468

470. Theorem. — *In two polar spherical triangles, each angle of one has the same measure as the supplement of the side of the other of which its vertex is the pole.*



Hypothesis. $\triangle ABC$ and $\triangle DEF$ are polar spherical triangles.

Conclusion. $\angle C$ has the same measure as the supplement of $\widehat{DE}$, $\angle F$ has the same measure as the supplement of $\widehat{AB}$, etc.

Proof. 1. $\triangle ABC$ and $\triangle DEF$ are polar triangles. Hyp.

2. Prolong $\widehat{CA}$ and $\widehat{CB}$, if necessary, to meet $\widehat{DE}$ at M and N, respectively.

3. Then $\angle C$ has the same measure as $\widehat{MN}$. § 461

4. $\widehat{DN} = 90°$ and $\widehat{ME} = 90°$. § 440

5. $\therefore \widehat{DN} + \widehat{ME} = 180°$. Ax. II

6. But $\widehat{DN} + \widehat{ME} = \widehat{MN} + \widehat{DE}$. Ax. XII

7. $\therefore \widehat{MN} + \widehat{DE} = 180°$, *i.e.* $\widehat{MN}$ and $\widehat{DE}$ are supp. Ax. I

8. $\therefore \angle C$ has the same measure as the supplement of $\widehat{DE}$. Ax. XII

9. Similarly, $\angle F$ has the same measure as the supplement of $\widehat{AB}$, etc.

471. Corollary 1. — *If a spherical triangle is equilateral, its polar triangle is equiangular; and conversely.*

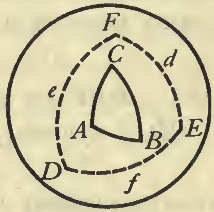
The proof is left to the student.

472. Definitions. — Two **mutually equiangular** triangles (See § 127) are triangles such that to each angle of one there corresponds an equal angle of the other. Two **mutually equilateral** triangles are triangles such that to each side of one there corresponds an equal side of the other.

473. Corollary 2. — *If two spherical triangles on the same sphere, or on equal spheres, are mutually equiangular, their polar triangles are mutually equilateral; and conversely.*

The proof is left to the student.

474. Theorem. — *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°.



Hypothesis. $\triangle ABC$ is any spherical triangle.

Conclusion. $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

Proof. 1. Let d, e, and f be the sides of the polar of $\triangle ABC$ of which A, B, and C, respectively, are the poles.

2. Then $\angle A$ has the same measure as $180° - d$,
$\angle B$ has the same measure as $180° - e$,
$\angle C$ has the same measure as $180° - f$. § 470

3. $\therefore \angle A + \angle B + \angle C$ has the same measure as
$540° - (d + e + f)$. Ax. III

4. But $d + e + f < 360°$ and $> 0°$. § 467

5. $\therefore 540° - (d + e + f) > 180°$ and $< 540°$. Subtraction

6. $\therefore \angle A + \angle B + \angle C > 180°$ and $< 540°$. Ax. XII

475. Corollary. — *A spherical triangle may have one, two, or three right angles. And it may have one, two, or three obtuse angles.*

The proof is left to the student.

476. Definitions. — A spherical triangle having two right angles is a **birectangular** spherical triangle. One having three right angles is a **trirectangular** spherical triangle.

The difference between the sum of the angles of any spherical triangle and 180° is the **spherical excess** of the triangle.

EXERCISES

1. The angles of a spherical triangle are 80°, 65°, and 135°. Find the sides of its polar triangle.

2. The sides of a spherical triangle are 120°, 84°, and 66°, respectively. How many degrees are there in each angle of its polar triangle?

3. A spherical triangle has two of its sides quadrants and the third side equal to 60°. Determine the angles of its polar triangle.

4. If two sides of a spherical triangle are quadrants, the triangle is birectangular.

5. If a spherical triangle is birectangular, the sides opposite the right angles are quadrants.

6. Three planes passed through the center of a sphere, each perpendicular to the other two, form on the spherical surface eight trirectangular triangles.

7. Find the sides of a spherical triangle if the angles of its polar triangle are 96° 37′ 36″, 87° 17′ 57″, 72° 46′ 32″.

8. What is the polar triangle of a spherical triangle all of whose sides are quadrants?

9. In any birectangular spherical triangle the side opposite the angle that is not a right angle has the same measure as that angle.

10. In any trirectangular spherical triangle each of the sides is a quadrant.

11. The angles of a spherical triangle are 80°, 90°, and 120°. What is the spherical excess?

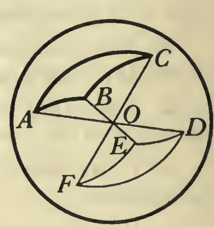
12. Any trirectangular spherical triangle coincides with its polar triangle.

13. Any exterior angle of a spherical triangle is less than the sum of the two opposite interior angles.

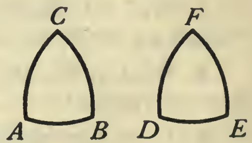
14. The arcs of the great circles which bisect the angles of a spherical triangle are concurrent.

15. Any arc of a great circle which is perpendicular to a second great circle passes through the pole of the latter.

477. Symmetrical spherical triangles. — Two triangles on the same sphere, or on equal spheres, are **symmetrical** when to each part of one there corresponds an equal part of the other, but the equal parts are arranged in *reverse order*.



It is easily seen that if three planes pass through the center of a sphere and do not intersect in one straight line, they cut the surface of the sphere in a pair of symmetrical spherical triangles, as $\triangle ABC$ and $\triangle DEF$ in the figure. Also it may be assumed that two symmetrical triangles of a sphere may be moved into such positions as are occupied by $\triangle ABC$ and $\triangle DEF$.

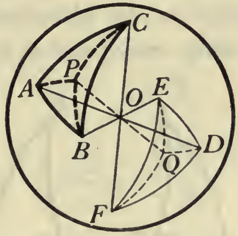


It is evident that, because of the curvature of the spherical surface, in general two symmetrical spherical triangles cannot be made to coincide, and hence are not congruent. See § 344.

If, however, two symmetrical spherical triangles are *isosceles*, as $\triangle ABC$ and $\triangle DEF$ in the figure, it is possible to make them coincide throughout. Hence it is inferred that:

If two symmetrical spherical triangles are isosceles, they are congruent.

478. Theorem. — *Two symmetrical spherical triangles have equal areas.*



Hypothesis. $\triangle ABC$ and $\triangle DEF$ are symmetrical spherical triangles.

Conclusion. $\triangle ABC = \triangle DEF$.

Proof. 1. $\triangle ABC$ and $\triangle DEF$ are symmetrical spherical triangles. Hyp.

2. $\therefore \triangle ABC$ and $\triangle DEF$ may be placed, as in the figure, so that AD, BE, and CF are diameters. § 477

3. Let P be the pole of the circle determined by points A, B, C, and let PQ be a diameter. Draw great circle arcs $\overset{\frown}{AP}$, $\overset{\frown}{BP}$, $\overset{\frown}{CP}$, $\overset{\frown}{DQ}$, $\overset{\frown}{EQ}$, $\overset{\frown}{FQ}$.

4. Then $\overset{\frown}{AP} = \overset{\frown}{BP} = \overset{\frown}{CP}$. § 438

5. $\overset{\frown}{AP} = \overset{\frown}{DQ}$, $\overset{\frown}{BP} = \overset{\frown}{EQ}$, $\overset{\frown}{CP} = \overset{\frown}{FQ}$. § 20 (V), § 173

6. $\therefore \overset{\frown}{DQ} = \overset{\frown}{EQ} = \overset{\frown}{FQ}$. Ax. I

7. $\therefore$ the symmetrical $\triangle$s APB and DQE are isosceles. § 464

8. $\therefore \triangle APB \cong \triangle DQE$. § 477

9. Similarly, $\triangle BPC \cong \triangle EQF$, $\triangle CPA \cong \triangle FQD$.

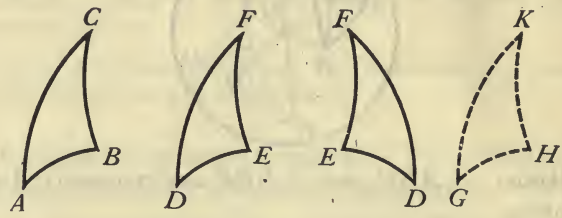
10. $\therefore \triangle ABC = \triangle DEF$. Ax. II

EXERCISE

Draw a figure for § 478 in which the pole P falls outside of $\triangle ABC$ and Q falls outside of $\triangle DEF$, and give the proof of the theorem.

SUGGESTION. — Each of the given triangles will be equivalent to the sum of two isosceles triangles minus a third one.

479. Theorem. — *If two spherical triangles on the same sphere or equal spheres have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are either congruent or symmetrical.*



Hypothesis. $\triangle ABC$ and $\triangle DEF$ are spherical triangles on the same sphere or equal spheres; $\overset{\frown}{AB} = \overset{\frown}{DE}$, $\overset{\frown}{AC} = \overset{\frown}{DF}$, $\angle A = \angle D$.

Conclusion. $\triangle ABC$ and $\triangle DEF$ are either congruent or symmetrical.

Suggestions. If the equal parts of the triangles occur in the same order, superpose $\triangle DEF$ upon $\triangle ABC$ so that $\angle D$ coincides with $\angle A$, and prove that the triangles coincide throughout.

If the equal parts occur in reverse order, construct $\triangle GHK$ symmetrical to $\triangle DEF$. Prove that $\triangle ABC \cong \triangle GHK$, and hence that $\triangle ABC$ is symmetrical to $\triangle DEF$.

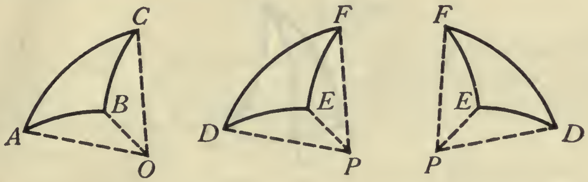
Write the proof in full.

480. Theorem. — *If two spherical triangles on the same sphere or equal spheres have a side and two adjacent angles of one equal respectively to a side and two adjacent angles of the other, they are either congruent or symmetrical.*

The proof is left to the student. Proceed as in the proof of § 479.

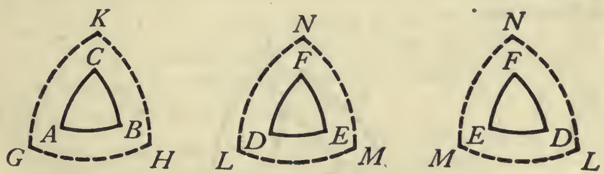
Write the proof in full.

481. Theorem. — *If two spherical triangles on the same sphere or equal spheres are mutually equilateral, they are either congruent or symmetrical.*



Left to the student. Prove diedral angles with edges AO and DP equal, and that $\angle A = \angle D$. Apply § 479.

482. Theorem. — *If two spherical triangles on the same sphere or equal spheres are mutually equiangular, they are either congruent or symmetrical.*

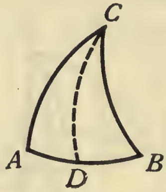


Hypothesis. $\triangle ABC$ and $\triangle DEF$ are triangles on the same sphere or equal spheres; $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

Conclusion. $\triangle ABC$ and $\triangle DEF$ are either congruent or symmetrical.

Suggestions. Draw $\triangle GHK$ and $\triangle LMN$ polars of $\triangle ABC$ and $\triangle DEF$. Prove $\triangle GHK$ and $\triangle LMN$ mutually equilateral. Then they are congruent or symmetrical. Why? Hence they are mutually equiangular. But $\triangle ABC$ and $\triangle DEF$ are polars of $\triangle GHK$ and $\triangle LMN$. Now, from these facts, by similar steps, show that $\triangle ABC$ and $\triangle DEF$ are congruent or symmetrical.

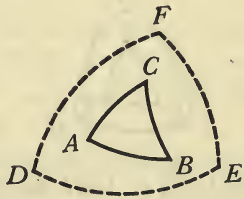
483. Theorem. — *In any isosceles spherical triangle the angles opposite the equal sides are equal.*



Draw a great circle arc from C to D, the middle point of $\widehat{AB}$. Compare $\triangle\ ADC$ and $\triangle\ DBC$ by § 481.

Write the proof in full.

484. Theorem. — *If two angles of a spherical triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.*



Hypothesis. In spherical $\triangle\ ABC$, $\angle\ A = \angle\ B$.

Conclusion. $\widehat{AC} = \widehat{BC}$.

Proof. 1. In spherical $\triangle\ ABC$, $\angle\ A = \angle\ B$. Hyp.

2. Let $\triangle\ DEF$ be the polar of $\triangle\ ABC$.

3. Then $\widehat{DF}$ and $\widehat{EF}$ have the same measures as the supplements of $\angle\ B$ and $\angle\ A$, respectively. § 470

4. $\therefore\ \widehat{DF} = \widehat{EF}$. Supp. of equals

5. $\therefore\ \angle\ D = \angle\ E$. § 483

6. But $\triangle\ ABC$ is the polar of $\triangle\ DEF$. § 469

7. $\therefore \overset{\frown}{AC}$ and $\overset{\frown}{BC}$ have the same measures as the supplements of $\angle E$ and $\angle D$, respectively. § 470

8. $\therefore \overset{\frown}{AC} = \overset{\frown}{BC}$. Supp. of equals

EXERCISES

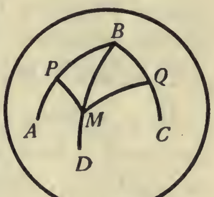
1. The great circle arc from the vertex of an isosceles spherical triangle to the middle point of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two triangles with equal areas.

2. Every point in the arc of a great circle perpendicular to another arc of a great circle at its middle point is equidistant from the ends of that arc.

3. Every point on the surface of a sphere that is equidistant from the ends of an arc of a great circle is on the arc perpendicular to that arc at its middle point.

4. What is the locus of points on the surface of a sphere that are equidistant from the ends of a given great circle arc?

5. If an arc of one great circle is perpendicular to an arc of another great circle at its middle point, any point without the former is unequally distant from the ends of the latter.



6. Any point on the bisector of a spherical angle is equally distant from the sides of the angle.

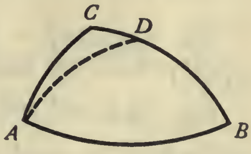
SUGGESTION. — Let BD bisect $\angle ABC$, and let MP be $\perp AB$. Mark off $BQ = BP$, and draw MQ. Prove that $MQ \perp BC$ and $MP = MQ$.

7. In any spherical triangle, if two angles are unequal, the opposite sides are unequal, the greater side being opposite the greater angle.

HYPOTHESIS. — In spherical $\triangle ABC$, $\angle BAC > \angle CBA$.

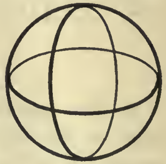
CONCLUSION. — $\overset{\frown}{BC} > \overset{\frown}{AC}$.

SUGGESTIONS. — Draw AD making $\angle BAD = \angle CBA$. Prove $\overset{\frown}{AD} = \overset{\frown}{BD}$. Then $\overset{\frown}{AD} + \overset{\frown}{DC} > \overset{\frown}{AC}$, etc.



8. State and prove the converse of Exercise 7.

485. Area of a trirectangular triangle. — If three planes are passed through the center of a sphere, each perpendicular to the other two planes, they form on the surface of the sphere eight *trirectangular triangles*, which are equal in area by § 482. Hence:



The area of a trirectangular triangle is equal to one eighth of the area of the surface of the sphere.

Such a triangle is sometimes used as a *unit of measure* of spherical surfaces.

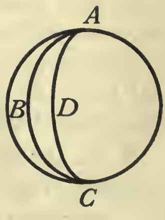
486. A spherical degree. — One of the 720 equal parts of the surface of a sphere is called a **spherical degree**.

A spherical degree is used as a unit of measure of spherical surfaces.

487. A lune. — A **lune** is that figure on the surface of a sphere which is formed by two great semicircles whose end points coincide. A lune, then, has two equal angles.

Thus, $ABCD$ is a lune whose equal angles are $\angle A$ and $\angle C$.

It is evident that lunes on the surface of the same sphere which have equal angles may be made to coincide, and hence are congruent. Consequently, it is inferred that:

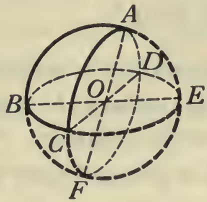


The area of a lune whose angle is n degrees is $\frac{n}{360}$ *of the area of the surface of the sphere.*

488. Corollary. — *The area of a lune, in spherical degrees, is equal to twice the number of degrees in the angle of the lune.*

For, by § 486 and § 487, the area of a lune whose angle is n degrees, is equal to $\frac{n}{360} \times 720$, or $2n$, spherical degrees.

489. Theorem. — *The area of a spherical triangle, in spherical degrees, is equal to the spherical excess of the triangle.*



Hypothesis. Spherical excess of $\triangle ABC = e$.

Conclusion. The number of sph. deg. in $\triangle ABC = e$.

Proof. 1. Spherical excess of $\triangle ABC = e$. Hyp.

2. Produce each side of $\triangle ABC$ to form the complete circle, and let AF, BE, and CD be the diameters in which their planes intersect.

3. The vertical angles at O are equal. § 20, V

4. $\therefore \overset{\frown}{BC} = \overset{\frown}{DE}$, $\overset{\frown}{BF} = \overset{\frown}{AE}$, $\overset{\frown}{CF} = \overset{\frown}{AD}$. § 173

5. $\therefore \triangle ADE$ and $\triangle BCF$ are either congruent or symmetrical. § 481

6. $\therefore \triangle ADE$ and $\triangle BCF$ are equal. Def. cong., § 478

7. Now lune $ABFC = \triangle ABC + \triangle BCF$,
lune $BCEA = \triangle ABC + \triangle ACE$, Ax. X
lune $CBDA = \triangle ABC + \triangle ABD$.

8. $\therefore ABFC + BCEA + CBDA =$
$2 \triangle ABC + (\triangle ABC + \triangle BCF + \triangle ACE + \triangle ABD)$. Ax. II

9. But $ABFC + BCEA + CBDA = 2\,(A + B + C)$ sph. deg. § 488

10. And $\triangle ABC + \triangle BCF + \triangle ACE + \triangle ABD =$ a hemispherical surface, or 360 sph. deg. § 486

11. $\therefore 2\,(A + B + C)$ sph. deg. $= 2 \triangle ABC + 360$ sph. deg. Ax. XII

12. $\therefore \triangle ABC = A + B + C - 180$ sph. deg. $= e$. Solving

490. Spherical excess of a spherical polygon. — The **spherical excess of a spherical polygon** is the difference between the sum of its angles and the sum of the angles of a plane polygon of the same number of sides.

491. Theorem. — *The area of a spherical polygon, in spherical degrees, is equal to the spherical excess of the polygon.*

The proof is left to the student. Draw diagonals from any vertex, dividing the polygon into triangles. How many triangles are there?

Write the proof in full.

EXERCISES

1. Find the area of a trirectangular triangle on a sphere whose diameter is 28 in.

2. Find the area of a lune whose angle is 30° on a sphere whose diameter is 20 in.

3. What part of the whole sphere is a triangle whose angles are 70°, 110°, and 120°?

4. What is the spherical excess of a triangle whose angles are 84° 27′ 20″, 96° 42′ 36″, and 116° 12′ 24″?

5. What is the area of a triangle whose angles are 80°, 92°, and 112°, on a sphere whose diameter is 36 in.?

6. What is the spherical excess of a hexagon whose angles are 84°, 167°, 140°, 106°, 98°, and 157°?

7. Find the area of a spherical pentagon whose angles are 120°, 140°, 155°, 158°, and 163°, on a sphere whose diameter is 10 in.

8. Find the area of that part of the earth's surface lying between the 75th and 90th meridians. Call the radius 4000 mi.

9. The areas of two lunes on the same sphere or on equal spheres are to each other as the angles of the lunes.

10. The areas of lunes having equal angles, but situated on unequal spheres, are to each other as the squares of the radii of the spheres on which they are situated.

MISCELLANEOUS EXERCISES

1. The radius of a sphere is 12 in. Find the area of a section made by a plane 8 in. from the center.

2. Find the locus of all points in space which are equidistant from two given points and at a given distance d from a third given point.

3. Any two vertical spherical angles are equal.

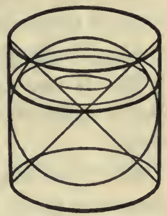
4. A right circular cylinder whose altitude is 8 in. is inscribed in a sphere whose radius is 6 in. Find the volume of the cylinder.

5. Any side of a spherical polygon is less than 180°.

6. If two adjacent sides of a spherical quadrilateral are greater, respectively, than the other two sides, the angle included by the two shorter sides is greater than the angle included by the two greater sides.

7. The radii of two concentric spheres are 8 in. and 12 in., respectively. A plane is tangent to the inner sphere. Find the area of the section of the outer sphere made by the plane. Find the area of the smaller zone of the outer sphere cut off by the plane.

8. A sphere is inscribed in a right circular cylinder, touching the lateral surface of the cylinder all the way around and tangent to both bases. Two cones have the bases of the cylinder as their bases and the center of the sphere as their common vertex. Any plane is passed through the figure parallel to the bases of the cylinder. Prove that the ring between the sections of the cylinder and cone is equal to the section of the sphere.



9. If two circles on a sphere have the same poles, their planes are parallel.

10. The line of centers of two intersecting spheres meets the surfaces of the spheres in the poles of their common circle.

11. Find the locus of the centers of all spheres tangent to a given plane at a given point.

12. Three spheres each of radius R are placed on a horizontal plane so that each is tangent to the other two. A fourth sphere of the same radius is placed on top of them so that it touches each. Find the distance from the highest point of the top sphere to the plane.

13. If a spherical triangle is isosceles, its polar triangle is also isosceles.

14. Prove that the volume of a hollow spherical shell whose outer and inner radii are R and r, respectively, is $\frac{4\pi(R-r)(R^2+Rr+r^2)}{3}$.

15. If a surveyor desires that the sum of the angles in any spherical triangle which he uses on the earth's surface shall be within one minute of 180°, what is the largest area that the triangle may inclose?

SUGGESTION. — See § 489.

16. The portion of a sphere bounded by the planes of two great semicircles and the lune which they form on the surface is called a *spherical wedge.* The angle between the planes of a spherical wedge is 15°, and the diameter of the sphere is 12 in. Find the volume of the wedge.

17. How many straight lines can be tangent to a sphere from a point outside of the sphere? Compare their lengths between the given point and the points of contact.

18. The points of contact of all lines tangent to a sphere from an external point lie in a circle.

19. The lines in Exercise 18 are the elements of a cone.

20. If two spheres are tangent to the same plane at the same point, the straight line joining their centers passes through the point of contact.

21. What is the locus of the centers of all spheres tangent to a given plane at a given point?

22. What is the locus of the centers of all spheres tangent to the faces of a diedral angle?

23. Find the edge of a cube inscribed in a sphere with radius 10 in. Find its area. Find its volume.

24. The volume of a polyedron circumscribed about a sphere is equal to the product of the area of its surface and one third of the radius of the sphere.

25. The sum of the arcs of great circles drawn from any point within a spherical triangle to the extremities of a side is less than the sum of the other two sides of the triangle.

26. If from a point within a spherical triangle arcs of great circles are drawn to the three vertices, their sum is less than the perimeter.

REFERENCES TO PLANE GEOMETRY

The following axioms, theorems, etc., which are given in the *Plane Geometry* are referred to in the proofs in the *Solid Geometry.* They are placed here for the convenience of the student in looking up the references.

AXIOMS

I. Things which are equal to the same thing, or to equal things, are equal to each other.

II. If equals are added to equals, the sums are equal.

III. If equals are subtracted from equals, the remainders are equal.

IV. If equals are multiplied by equals, the products are equal.

V. If equals are divided by equals, the quotients are equal.

VI. Like powers, or like positive roots, of equals are equal.

VII. If equals are added to or subtracted from unequals, or if unequals are multiplied or divided by the same positive number, the results are unequal in the same order.

VIII. If unequals are subtracted from equals, the remainders are unequal in the reverse order.

IX. If unequals are added to unequals in the same order, the sums are unequal in that order.

X. The whole of a thing is equal to the sum of all of its parts, and is greater than any one of its parts.

XI. If the first of three things is greater than the second, and the second greater than the third, then the first is greater than the third.

XII. A quantity may be substituted for its equal in any expression.

DEFINITIONS, THEOREMS, ETC.

§ 6. A *straight line* is represented by placing a ruler, or straightedge, upon a plane surface and marking along the edge of the ruler.

§ 13. Two angles are *equal* if, and only if, they may be made to coincide throughout. If an angle is placed upon an equal angle so that the vertices and a pair of sides coincide, the other sides must coincide.

§ 17. A *right angle* is one half of a straight angle.

§ 20. I. Any two straight angles are equal.

II. Any two right angles are equal.

III. The complements of equal angles are equal.

IV. The supplements of equal angles are equal.

V. Vertical angles are equal.

§ 26. If two parallel lines are cut by a transversal, the corresponding angles are equal.

§ 34. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

§ 41. Two straight lines perpendicular to the same straight line are parallel.

§ 46. Only one straight line can be drawn through a given point parallel to a given straight line.

§ 48. The sum of the angles of any triangle is equal to a straight angle.

§ 53. At a given point on a given line only one perpendicular can be drawn to the line.

§ 54. Only one perpendicular to a given line can be drawn through a given point not on the line.

§ 55. Two straight lines perpendicular respectively to two intersecting straight lines must meet.

§ 60. If one of two figures may be placed upon the other so that they coincide throughout, the figures are called *congruent.*

§ 63. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.

§ 64. Two right triangles which have the legs of one equal respectively to the legs of the other are congruent.

§ 67. Two right triangles are congruent if a leg and an acute angle of one are equal respectively to a leg and an acute angle of the other.

§ 68. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.

§ 76. If the three sides of one triangle are equal respectively to the three sides of another, the triangles are congruent.

§ 82. The opposite sides of a parallelogram are equal.

§ 83. A diagonal divides a parallelogram into two congruent triangles.

§ 85. Two parallel lines are everywhere equidistant.

§ 90. If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

§ 100. If a line-segment is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, then it bisects the other also and is equal to one half of the sum of the bases.

§ 101. The sum of the interior angles of a convex polygon of n sides is $(n - 2)$ straight angles.

§ 105. The locus of points equidistant from the ends of a line-segment is the perpendicular bisector of it.

§ 107. Two points each equidistant from the ends of a line-segment determine the perpendicular bisector of it.

§ 114. The medians of any triangle are concurrent at a point of trisection of each.

§ 118. (1) If four numbers are in proportion, the product of the extremes equals the product of the means.

(2) If the product of two numbers equals the product of two other numbers, the four numbers are in proportion, one pair of factors being extremes and the other pair means.

(3) If four numbers are in proportion, they are in proportion by inversion.

(4) In any proportion, the means may be interchanged, or the extremes interchanged, without destroying the proportion.

(5) The terms of any proportion are in proportion by addition.

(6) The terms of any proportion are in proportion by subtraction.

(7) Like powers or like roots of the terms of a proportion are in proportion.

(8) If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

§ 123. If DE is parallel to side AB of triangle ABC and meets AC at D and BC at E, then

$$\frac{AC}{AD}=\frac{BC}{BE} \text{ and } \frac{AC}{DC}=\frac{BC}{EC}.$$

§ 127. Two polygons which have the angles of one equal respectively to the angles of the other, taken in order, are called *mutually equiangular*.

Similar polygons are those which (1) are mutually equiangular and (2) have their corresponding sides proportional.

§ 128. If two triangles are mutually equiangular, they are similar.

§ 129. If two triangles have their corresponding sides proportional, the triangles are similar.

§ 130. If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, they are similar.

§ 146. The sum of any two sides of a triangle is greater than the third side.

§ 148. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the angles opposite these sides are unequal, the greater angle being opposite the greater side.

§ 151. (1) A diameter of a circle equals two times a radius.

(2) Radii of the same circle or equal circles are equal.

(3) If the radii of two circles are equal, the circles are equal.

(4) In the same or equal circles, equal central angles intercept equal arcs.

(5) In the same or equal circles, equal arcs subtend equal central angles.

(6) In the same or equal circles, equal arcs are subtended by equal chords.

(7) In the same or equal circles, equal chords subtend equal arcs.

(8) A point is at a distance from the center of a circle equal to, greater

than, or less than the radius, according as it is on, outside of, or within the circle; and conversely.

§ 156. A line through the center of a circle perpendicular to a chord bisects the chord and the subtended arcs.

§ 164. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

§ 173. A central angle has the same numerical measure as its intercepted arc.

§ 185. If, in geometric constructions, the instruments used are limited to the ungraduated straightedge and compasses alone, there are some constructions which cannot be made. There are three such impossible constructions which have interested mathematicians since the time of the ancient Greeks:

(1) To trisect (cut into three equal parts) any given angle.

(2) To construct a square which shall have the same area as a given circle.

(3) To construct a cube which shall have twice the volume of a given cube.

§ 196. In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

§ 200. If two chords intersect, the product of the segments of one is equal to the product of the segments of the other.

§ 215. Parallelograms having equal altitudes and equal bases are equal.

§ 228. Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including those angles.

§ 274. (1) If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem approaches the radius of the circle as a limit.

(2) If the number of sides of a regular inscribed polygon is indefinitely increased, the perimeter approaches the circumference of the circle as a limit.

(3) If the number of sides of a regular circumscribed polygon is indefinitely increased, the perimeter of the polygon approaches the circumference of the circle as a limit.

(4) If the number of sides of a regular circumscribed polygon is indefinitely increased, the area of the polygon approaches the area of the circle as a limit.

§ 275. (1) If two variables are equal and each approaches a limit, the limits are equal.

(2) If the limit of a variable x is a, then the limit of kx is ka, where k is any constant.

(3) If the limit of a variable x is a, then the limit of $\frac{x}{k}$ is $\frac{a}{k}$, where k is any constant.

§ 281. The area of a circle equals πR^2.

§ 284. The area of a sector of a circle is equal to one half of the product of its radius and its arc.

SOLID GEOMETRY

INDEX OF DEFINITIONS